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Supersonic Aerodynamics—Principles and Applications

The Tenth Wright Brothers Lecture*

THEODORE VON KÁRMÁN

(1) Velocity of Sound, Propagation of Pressure

The first theoretical computation of sound velocity was given by Isaac Newton in his *Principles of Natural Philosophy*. He found that the velocity of propagation of a pressure pulse is directly proportional to the square root of the elastic force resisting the compression of air and inversely proportional to the square root of the density of the medium. Carrying out the calculation, he obtained a value of 979 ft. per sec. for sound velocity in air at sea level under standard conditions and found that this value is about 15 per cent lower than the experimental value of 1,142 ft. per sec. deduced from gunshot observations. He explained the discrepancy by the presence of suspended solid particles and water vapor in the atmosphere. Later, Laplace found that Newton's method of calculation of the elastic force is equivalent to the assumption of isothermal compression, whereas the true process is very nearly adiabatic. Newton, of course, could not foresee the thermodynamic relations that were unknown at his time; nevertheless, it is interesting to see that even such a genius could succumb to the temptation of explaining an essential discrepancy between theory and experiment by wishful thinking.

Until recently, the study of the motion of bodies having velocity faster than sound belonged to the realm of ballistics. As a matter of fact, when the science of modern aerodynamics was developed a few decades ago, most theories were based on the assumption that the air could be considered as an incompressible fluid. It was shown that the error in the computation of air forces produced by the motion of an airplane is about one-half times the square of the ratio of flight velocity to sound velocity. This ratio is known as Mach Number, in honor of the Viennese physicist and philosopher, to whom we owe many findings on high-speed flow and beautiful optical methods of observation. If the flight velocity is 150 m.p.h., the error is about $\frac{1}{2}(\frac{1}{5})^2 = 2$ per cent. As the flight velocity of airplanes increased, it became necessary to consider the so-called "compressibility effects."

I believe we have now arrived at the stage where knowledge of supersonic aerodynamics should be considered by the aeronautical engineer as a necessary prerequisite to his art. This branch of aerodynamics should cease to be a collection of mathematical formulas and half-digested, isolated, experimental results. The aeronautical engineer should start to get the same feeling for the facts of supersonic flight as he acquired in the domain of subsonic velocities by a long process of theoretical study, experimental research, and flight experience.

(2) The Three Rules of Supersonic Aerodynamics

The following rules are based on the assumption that the disturbance caused in the air by a moving body can be considered slight. The influence of finite disturbances will be discussed in Section 10.

(a) *The Rule of Forbidden Signals*

Since a slight pressure change is propagated at sound velocity, it is evident that the effect of pressure changes produced in the air by a body moving at a speed faster than sound cannot reach points ahead of the body. It may be said that the body is unable to send signals ahead. It is seen that there is a fundamental difference between subsonic and supersonic motion. Consider the case of subsonic stationary motion—for example,

* Presented before the Institute of the Aeronautical Sciences in the U.S. Chamber of Commerce Auditorium, Washington, D.C. Received April 18, 1947.

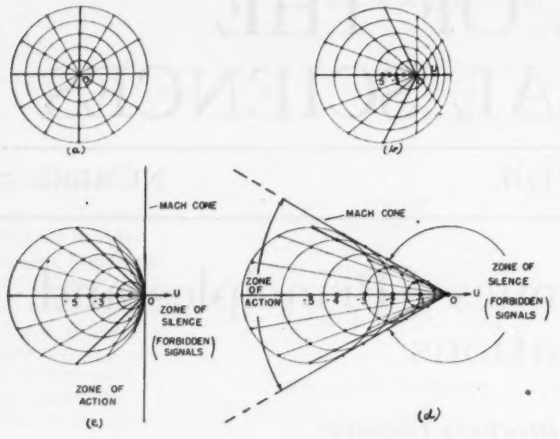


FIG. 1. Point source moving in compressible fluid. (a) Stationary source. (b) Source moving at half the speed of sound. (c) Source moving at the speed of sound. (d) Source moving at twice the speed of sound.

the uniform level flight of an airplane. Then a pressure signal travels ahead at sound velocity minus flight velocity relative to the airplane, whereas a signal travels backward at a speed equal to the sum of the flight and sound velocity. So the distribution of the effects is no longer symmetric; nevertheless, every point in space is reached by a signal, provided the flight started from an infinitely remote point. (For this consideration we neglect viscosity—i.e., the absorption of energy in the air.) As can easily be seen, this is not the case in supersonic flight, and one obtains the second rule, which refers to the zone of action and the zone of silence.

(*b*) *Zone of Action and Zone of Silence*

Consider the simplest case of a point source (Fig. 1). Fig. 1a shows the spherical surfaces that the pressure effect reaches in equal time intervals in the case of a point source at rest. Fig. 1b shows the same surfaces relative to the point source moving with a speed less than that of sound. Fig. 1c represents the case of a point source moving with sonic velocity, and Fig. 1d, the case of a source moving faster than sound. It is seen that in the last case all action is restricted to the interior of a cone that includes all spheres. The outside of this cone can be called the zone of silence. It is easily seen that the trigonometrical sine of the half vertex angle of the cone is equal to the reciprocal of the Mach Number. This angle is called the Mach angle. The cone that separates the zone of action from the zone of silence is called the Mach cone.

According to this rule, a stationary point source in a supersonic stream produces action only on points that lie on or inside the Mach cone extending downstream from the point source. Conversely, pressure and velocity at an arbitrary point of the stream can be influenced only by disturbances acting at points that lie on or inside a cone of the same vertex angle extending upstream from the point considered.

(*c*) *The Rule of Concentrated Action*

This rule expresses another characteristic difference between subsonic and supersonic motion. It concerns the distribution of the pressure effect in space relative to the moving body. The points plotted in Figs. 1a to 1d show the location of mass points that are supposed to be emitted from the source and move at sound velocity in all directions. They illustrate qualitatively the distribution of the density of action in the various cases. In the subsonic case one finds that the pressure effect not only decreases with increasing distance from the source but is also dispersed in all directions. In the case of a body moving at supersonic velocity, the bulk of the effect is concentrated in the neighborhood of the Mach cone that forms the outer limit of the zone of action. This phenomenon is also easily seen in the examples to be considered in the following sections.

(3) The Mechanism of Drag

There are two concepts of the drag of a body moving in a fluid medium. First, one may consider the pressure and friction forces acting on the surface of the body and determine their resultant. Second, one may consider the body and a suitably bounded region of the surrounding air as one mechanical system and compute or measure the normal and tangential stresses acting on the boundaries of this system, called control surfaces, plus the flow of momentum due to the fluid entering or leaving the system through the same boundaries. The engineer may look at the first concept as a more direct one. Often, the second concept is more practical, however, as shown, for example, by its successful application as an experimental method. It is commonly known that the best determination of the profile drag of an airfoil section is achieved by the so-called wake method, which is precisely an application of the second concept. Also, in the age of propulsion by reaction, every engineer should be acquainted with the definition of forces acting on a moving body by action and reaction.

According to the so-called paradox of d'Alembert, the motion of a body in a nonviscous incompressible fluid does not involve drag if the motion does not produce eddies (vorticity) and the flow does not separate from the body. This statement is based on the existence of a vortex-free motion around the body, in which all disturbances completely disappear at infinity. The resistance in an actual fluid is then due to friction forces and flow separation. The loss of momentum equivalent to the drag can be found in the wake that follows the body. This statement concerning the relation between wake and drag is also correct for the motion of the body in a compressible fluid if the motion is subsonic. Moreover, the drag is zero—i.e., the d'Alembert theorem

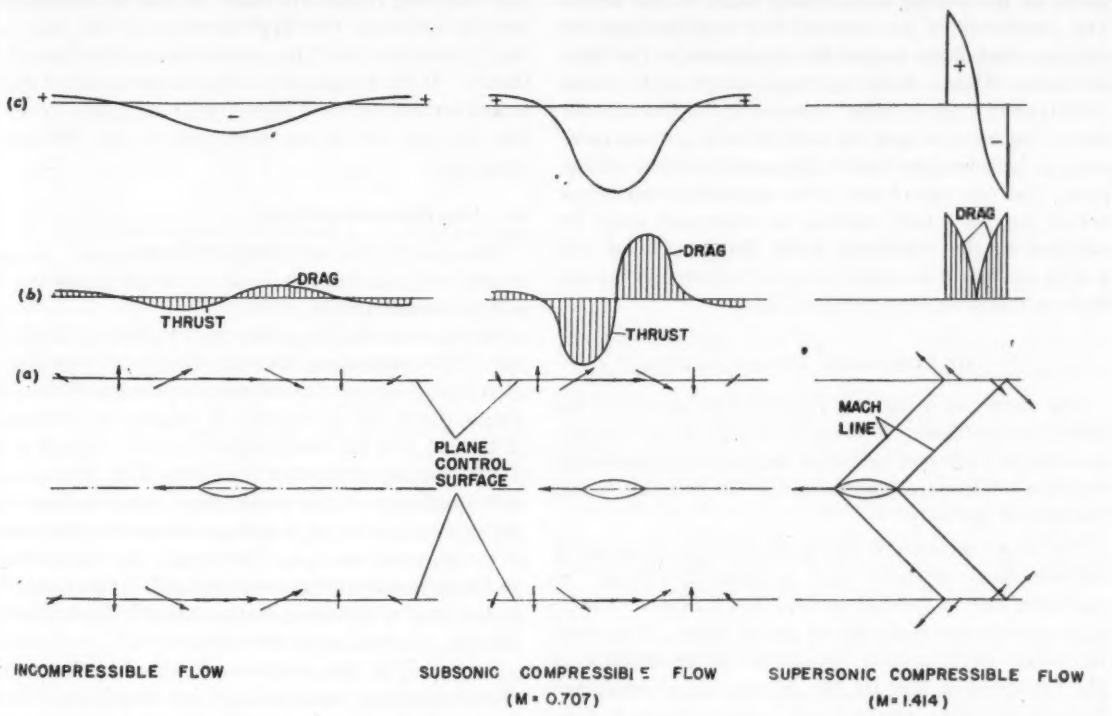


FIG. 2. Two-dimensional symmetric airfoil in flight. (a) Velocities through control surface. (b) Horizontal momentum transfer through control surface. (c) Pressure at control surface.

holds in a nonviscous compressible fluid, provided a vortex-free continuous motion exists around the body. Whereas, however, in the case of an incompressible fluid, the equations of flow always admit such a solution, in the case of a compressible fluid this is true only below a certain critical Mach Number that is less than unity. Between this Mach Number and Mach Number 1, the wake is due not only to friction and flow separation but, as will be shown in a later section, also to the existence of shock waves. Hence, although the mechanism of the wake formation may be more complex in this case, nevertheless the total loss of momentum equivalent to the drag always appears in the wake as long as the body moves at subsonic velocity.

If the body moves at supersonic velocity, a new type of drag appears. For convenience, consider the air nonviscous and assume that the motion of the body produces disturbances that can be considered small. At a certain distance from the moving body this second assumption will, in general, be satisfied. Consider, now, the body and the surrounding air inside of a cylindrical control surface as one mechanical system. Then one finds that, because of the concentrated action that characterizes the pressure propagation from a source moving at supersonic velocity, the total flux of momentum of the air masses entering and leaving the cylindrical boundary remains finite even when the boundary is removed to an arbitrarily large distance.

Fig. 2 refers to the case of a two-dimensional symmetric airfoil, with sharp leading edge, moving through air initially at rest. Let us consider the flow through a plane parallel to the plane of symmetry at a certain distance from the body. The diagram shows the distribution of induced velocities and the horizontal component of momentum transfer along this plane for three cases. It is evident that the reaction of outgoing flow having a horizontal component opposite to the flight direction and incoming flow with a horizontal component in the flight direction is equivalent to a propulsive thrust acting on the body. Conversely, outgoing flow with a component in the flight direction and incoming flow directed opposite to the flight direction give rise to a drag. In the two subsonic cases ($M \sim 0$ and $M = 0.707$), thrust and drag contributions are balanced, and the total horizontal momentum transfer is equal to zero.

The influence of increasing Mach Number is essentially increased magnitude of the induced velocities and the increasing concentration of the disturbance in the region extending laterally outward from the body. The increasing concentration of action is also illustrated by the pressure distribution on the control surface. In the supersonic case ($M = 1.414$), the disturbance is restricted to two strips bounded by two Mach lines. These lines are the intersections of planes that are envelopes of the Mach cones starting from

points of the leading and trailing edges of the airfoil. The component of the outward flow is in the flight direction; that of the inward flow is opposite to the flight direction. Hence, both represent drag on the body. This type of drag is called "wave drag." The calculation of the wave drag is the first of the important problems to be solved by supersonic aerodynamics. It appears that one can obtain a fair approximation to the actual drag of a body moving at supersonic speed by addition of the calculated wave drag coefficient and a drag coefficient corresponding to friction and separation, extrapolated from subsonic data.

(4) The Linearized Theory of Drag

The theory of wave drag requires the solution of the following problem: A body is placed in an initially uniform and parallel airstream moving with supersonic velocity. What are the changes in the flow due to the presence of the body?

The exact solution of this problem requires the use of mathematical methods that in principle should be sufficient but in general involve much labor. Hence, approximate methods are of great value. The most important simplification consists of the linearization of the equations of flow by the assumption of small perturbations, more precisely of velocity perturbations that are small in comparison to both flight velocity and sound velocity. This theory gives a fair approximation for the drag of slender or flat bodies with pointed noses or sharp leading edges. Fortunately, most parts of future supersonic airplanes or missiles will necessarily have such geometrical shapes.

The linearized theory can be carried out with relatively simple analytical means, since the linearized equations of the flow are essentially identical with the equations for wave motion of small amplitude. Thus, many known methods of the wave theory can be applied to such simplified supersonic aerodynamics. This is especially true in the case of slender bodies of revolution (for example, a fuselage of an airplane or the main body of a missile) and of flat bodies, like airplane wings. In these cases a further step of simplification can be made concerning the boundary conditions of the flow problem—i.e., the requirement that the air follow the surface of the body. This condition determines, in the case of axially symmetric flow, the direction of the velocity vector at the surface and, in the case of a flat body, the direction of the component of the velocity vector that lies in the plane normal to the center surface. It is possible to solve the linearized differential equations and satisfy these boundary conditions exactly, but, in general, numerical or graphical methods must be applied. Hence, it is desirable to simplify the problem further by replacing the exact boundary conditions by conditions that, by a limiting process, are referred to the axis of the body of revolution or the plane plan form of the wing instead of the actual surface. The following results are based on this approximation. Strictly speaking, this approximation is the only one that is consistent with the assumptions of the linearized theory. If the boundary conditions are satisfied at the actual surface, terms of higher order are generally taken into account which are neglected in the differential equations.

(*a*) *Two-Dimensional Flow*

This case is that of a wing of infinite span. A symmetric wing section is assumed; wings producing lift will be discussed later.

In this case the linearized theory yields a simple result. The supersonic flow of velocity U produces at each surface element of the wing a pressure of the magnitude $(2\delta/\sqrt{M^2-1})\ (\rho U^2/2)$, where ρ is the density of the air, δ is the local angle of attack, and M is the Mach Number of the flow (i.e., of flight). The remarkable simplicity of this result arises from the fact that the pressure acting at a surface element is independent of the shape of the rest of the section and depends only on the inclination of the element itself. It is known that in the case of subsonic motion there is an interaction between all elements of the surface.

According to this simple result, the drag of a section of unit length in span direction can be expressed as the product of the pressure $\rho U^2/2$, the chord c, and a drag coefficient of the magnitude $C_D = 4\bar{\delta}^2/\sqrt{M^2-1}$, where $\bar{\delta}^2$ is the mean square of the angles of inclination occurring at the surface of the section. For a double wedge section, $\bar{\delta}^2$ is equal to the square of the thickness-chord ratio. Thus, for example, for a 6 per cent double wedge section and for Mach Number $\sqrt{2}$, the wave drag coefficient is 0.0144, about twice the coefficient of the profile drag of a good subsonic section at low Mach Number.

In the case of two-dimensional flow, pressure effects are restricted to two strips whose angles of inclination with respect to the flow direction are equal to the Mach angle. With the approximation used, compressions and expansions are propagated with unchanged intensity along the Mach lines. As was mentioned previously, the transfer of momentum is also restricted to these strips. By calculating the action and reaction of the air passing through lateral surfaces, the preceding result concerning drag can easily be confirmed.

(*b*) *Body of Revolution*

One of the best-known methods for construction of an incompressible flow around a body of revolution uses the concepts of sources and sinks. These concepts can be carried over in the approximate theory of compressible flow, both subsonic and supersonic. The formula for the velocity potential of the flow produced in an incompressible fluid by a source located on the x-axis at the point $x = \xi$ is

$$\varphi = (Q/4\pi)\ [1/\sqrt{(x-\xi)^2 + r^2}] \qquad (4.1)$$

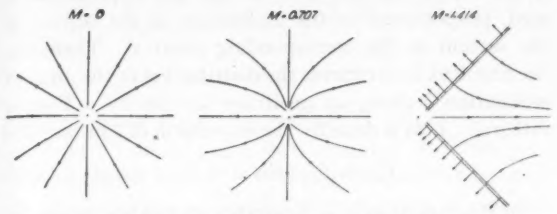


FIG. 3. Streamlines of source flows used in the theory of compressible fluids.

in which x and r are cylindrical coordinates. Q is called the intensity of the source and is the total volume of fluid emitted by the source in unit time. This formula can be generalized by writing

$$\varphi = A/\sqrt{(x-\xi)^2 - (M^2-1)r^2} \tag{4.2}$$

The function φ, given by Eq. (4.2), is a solution of the linearized flow equations for arbitrary Mach Number M of the main flow. Fig. 3 shows schematically a source defined by Eq. (4.2) for the three cases $M \to 0$, $M < 1$, and $M > 1$. In accordance with the rules of the forbidden signals and zone of action and zone of silence, in the supersonic case the flow is restricted to the inside of the Mach cone. In fact, Eq. (4.2) gives real values for φ only in the interior of the Mach cone. The flow velocity is infinite along the surface of the cone itself. For this reason, no concentrated sources and sinks can be used in the supersonic theory. One has to construct line sources with continuously distributed intensity. Accordingly the application of this theory is restricted to body shapes with pointed nose and pointed tail. If the simplified boundary conditions are used, sharp corners in the meridian section must also be excluded.

With the simplified boundary conditions the theory yields the following results:

(1) The intensity of the source distribution along the axis of the body, $f(\xi)$, defined as the volume of fluid per unit time per unit length of axis emitted at the location ξ on the axis, is given by

$$f(\xi) = U(dS/dx) \tag{4.3}$$

where U is the free stream velocity and S is the cross-sectional area of the body.

(2) The potential of the source distribution representing the body is

$$\varphi = \frac{1}{2\pi}\int_0^l \frac{f(\xi)d\xi}{\sqrt{(x-\xi)^2 - (M^2-1)r^2}} \tag{4.4}$$

in which x and ξ are length coordinates along the axis and l is the length of the body.

(3) The wave drag of the body is given by the formula

$$D_w = -\frac{\rho}{4\pi}\int_0^l \int_0^l dx\, d\xi\, f'(x)f'(\xi) \log|x-\xi| \tag{4.5}$$

where $f'(x) = df/dx$ and ρ is the density of air in the undisturbed flow.

It will be noted that there is a remarkable analogy between the wave drag of a slender body of revolution and the induced drag of a loaded line. In fact, if the function $f(x)$ denotes the spanwise distribution of the circulation on a loaded line, the induced drag is given by the equation:

$$D_i = -\frac{\rho}{4\pi}\int_0^l \int_0^l dx\, d\xi\, f(x')f'(\xi) \log|x-\xi| \tag{4.6}$$

which is identical with Eq. (4.5). This analogy is useful to the aeronautical engineer who is well acquainted with the concept and theory of the induced drag.

The distribution of induced velocities and horizontal momentum transfer produced by a slender body in supersonic flow is shown in Fig. 4. It is noted that in this case the disturbance caused by the body extends into the interior of the Mach cone starting from the rear end of the body. We recall that in the two-dimensional case the effect is confined to a strip.

(c) *Airplane Wing with Arbitrary Plan Form and Thin Symmetrical Sections*

The theory for this problem can be built up by using the concept of sources and sinks distributed continuously over the center-plane of the wing. One finds that in this case the surface density of the source of distribution is proportional to the angle of inclination of the wing surface measured in a vertical plane erected

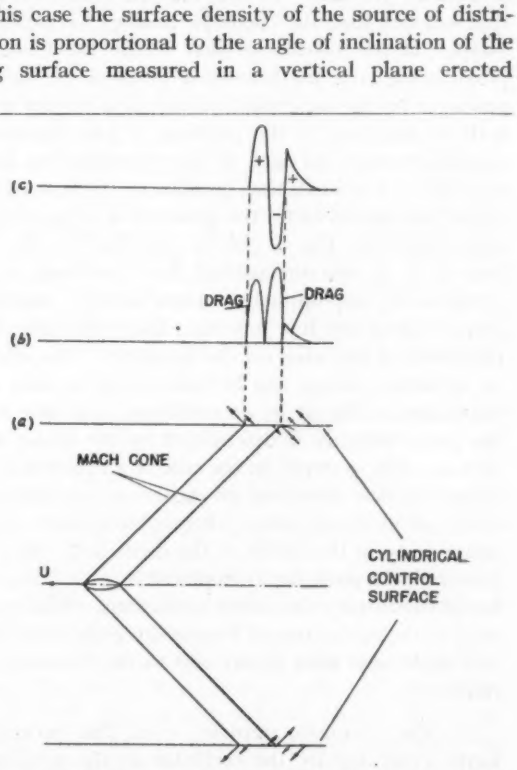


FIG. 4. Body of revolution in supersonic flight. (a) Velocities at the control surface. (b) Transfer of horizontal momentum. (c) Pressure distribution at the control surface.

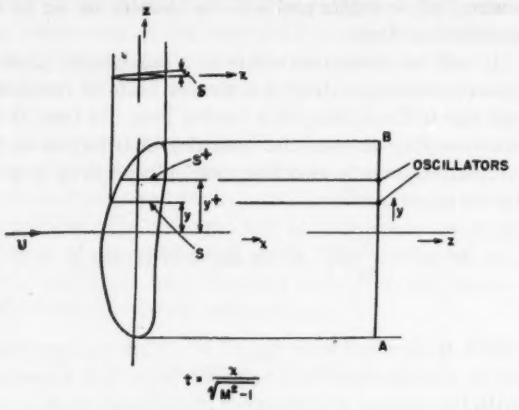


FIG. 5. Schematic representation of a flat body with symmetric sections. Acoustic analogy.

through the direction of flight. The pressure distribution over the wing and the total value of the wave drag can be calculated by the summation of the actions of all the sources.

R. T. Jones found useful methods for carrying out such summations, especially for wings with sweepback and taper. He uses appropriately chosen oblique coordinates for this purpose.

Another promising method is given by the concept of Fourier analysis. In this respect, a consideration that can be called the "acoustic analogy" is extremely useful. It was mentioned that within the linear approximation, the mathematical problem of the flow produced by the supersonic motion of a slender or flat body is identical to the problem of two-dimensional acoustic waves. In fact, if one considers the length $x/\sqrt{M^2-1} = t$ as a time coordinate, the three-dimensional flow produced by the presence of a thin airplane wing shown in Fig. 5 can be described by the time history of a two-dimensional flow produced in the y,z-plane by appropriately chosen acoustic oscillators located along the line AB—i.e., the projection of the plan form of the wing on the y,z-plane. The effect of an arbitrary section can be replaced by a pulse communicated to the air by an oscillator; the time law of the pulse intensity is determined by the shape of the section. For example, in the case of a cylindrical wing normal to flow direction, all oscillators simultaneously carry out identical pulses, whereas sweepback means a phase delay in the action of the oscillators. Now it is known that a pulse can be replaced by an infinite number of elementary sinusoidal oscillations. This concept leads to the application of Fourier integrals in the linearized supersonic wing theory and to the following main results:

(1) The "acoustic impulse"—i.e., the vertical velocity generated by the oscillator in the y,z-plane at any instant t—is proportional to the vertical velocity in the three-dimensional flow produced by the presence of a section, and therefore within the approximation used, proportional to the inclination of the surface of the section at the corresponding point x. Therefore, the first task is to express the distribution of the angle of inclination δ along an arbitrary section by a Fourier integral. This is done by the expression of δ in the form

$$\delta = a\int_0^\infty d\nu[f_1(\nu)\sin \nu t + f_2(\nu)\cos \nu t] \qquad (4.7)$$

In this equation, a is a suitably chosen length parameter—for example, the half chord of one master section. The parameter ν, which is used as variable of integration, is inversely proportional to the wave length in the three-dimensional flow problem and proportional to the frequency of the oscillators in the acoustic analogy. The functions $f_1(\nu)$ and $f_2(\nu)$ give the amplitudes of the sine and cosine components of the Fourier analysis. With the exception of an infinite normal wing with constant section, f_1 and f_2 are also functions of the coordinate y (spanwise location); outside the span, $f_1 = f_2 = 0$.

(2) The wave drag of the wing appears in the acoustic analogy as the energy transmitted to infinity during the entire period of the process. Since one has to consider an infinite number of oscillators distributed along the span, or, more correctly, a chord oscillating in the two-dimensional y,z-plane, one has to compute the mutual interaction of the oscillators or the elements of the chord.

Consider two wing sections, S and S^* (Fig. 5), at a distance $|y - y^*|$, measured spanwise, and replace these sections by oscillators. The amplitudes of the sine and cosine components as functions of the frequency are given by $f_1(\nu), f_2(\nu)$ and $f^*_1(\nu), f^*_2(\nu)$, respectively. Then it can be shown that the contribution of the two sections to the energy transmitted to infinity is proportional to the quantity

$$\int_0^\infty (f_1 f^*_1 + f_2 f^*_2)\nu J_0(\nu|y - y^*|)d\nu$$

where ν is the frequency parameter and J_0 denotes the Bessel function of zero order. This rather simple result indicates a straightforward way for computation of the wave drag of wings with various plan forms. The equation for the total wave drag can be written in the form:

$$D = \pi\rho U^2 a^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dy\, dy^* \int_0^\infty \nu d\nu (f_1 f^*_1 + f_2 f^*_2) J_0(\nu|y - y^*|) \qquad (4.8)$$

(5) Some Results of Wave Drag Calculations

(a) *The Wave Drag of Normal Straight Wings with Constant Airfoil Section*

The results for infinite aspect ratio were given in the foregoing section.

It is evident from the rule of the forbidden signals that the influence of the tips of a wing with finite aspect ratio is restricted to the inside of the Mach cones starting from the tip fronts.

The theory shows that the drag coefficient C_D can be expressed in the form:

$$C_D = C_{D_0}\,\varphi(\text{A.R.}\sqrt{M^2-1})$$

where C_{D_0} is the wave drag coefficient of the two-dimensional airfoil of the same section and A.R. is the geometrical aspect ratio. The parameter $\text{A.R.}\sqrt{M^2-1}$ can be considered as the effective aspect ratio parameter. It is equal to the ratio of the span to the length c^* shown in Fig. 6, which is actually the portion of the trailing edge within the Mach cone starting from the tip of the leading edge. The function φ is equal to unity when $\text{A.R.}\sqrt{M^2-1} \geq 1$ and decreases with the aspect ratio when $\text{A.R.}\sqrt{M^2-1} < 1$. Hence, the aspect ratio influences the average drag coefficient only if the entire trailing edge is within both Mach cones, which start from the two tips of the leading edge.

Fig. 6 shows the function φ for the special case of a wing with double wedge section. The distribution of the sectional wave drag coefficient C_d is shown in Fig. 7 for $\text{A.R.}\sqrt{M^2-1} = 2$. The sectional wave drag coefficient C_d is connected with C_D by the relation $C_D = (1/b)\int_0^b C_d dy$, where b is the span and y is measured along the span. At either tip C_d is only one-half of the two-dimensional value. For a rectangular wing of geometrical aspect ratio 2, this figure corresponds to $M = \sqrt{2}$. For a wing of geometrical aspect ratio 8, it corresponds to $M = \sqrt{1.0625}$. For $\text{A.R.}\sqrt{M^2-1} > 2$, the drag distribution along each portion of the tip length equal to c^* is identical to that shown in the diagram, whereas for the portion of the wing between the tips $C_D/C_{D_0} = 1$. If properly used, this simple diagram furnishes the drag distribution for wings of any aspect ratio at any Mach Number larger than unity.

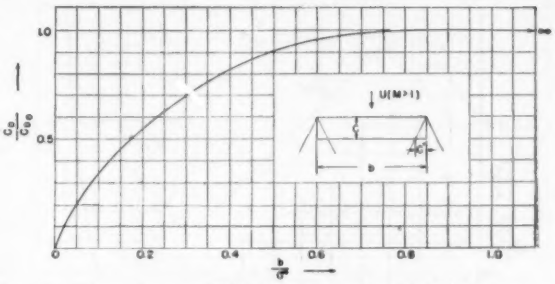


FIG. 6. Wave drag coefficient of normal straight wings as a function of the aspect ratio.

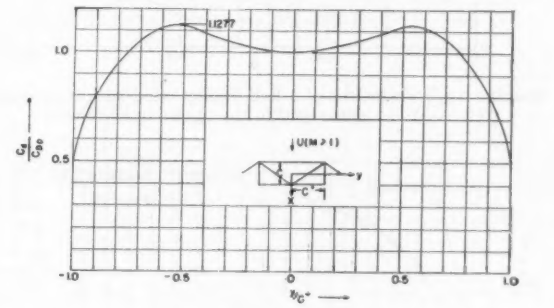


FIG. 7. Spanwise distribution of the sectional wave drag coefficient of a normal wing ($\text{A.R.}\sqrt{M^2-1} = 2$).

(b) *Infinite Uniform Wing with Sweepback*

The pressures acting on an infinite uniform wing with sweepback depend only on the component of the main flow normal to the wing axis. This is made evident by the consideration that in a nonviscous fluid the velocity component parallel to the wing axis is undisturbed by the presence of the wing and therefore cannot have any influence on pressure and drag.

One concludes that the wave drag of a uniform wing with infinite span is zero if the sweepback angle is so large that the velocity normal to the wing axis becomes subsonic.

This occurs when $\tan\gamma > \sqrt{M^2-1}$, where γ is the angle of sweepback. The ratio $\beta = \tan\gamma/\sqrt{M^2-1}$ is one of the fundamental parameters of supersonic aerodynamics. It may be called the effective sweepback parameter. When $\beta > 1$, the velocity normal to the wing is subsonic; when $\beta < 1$, it is supersonic.

It follows from elementary considerations that, for $\beta < 1$, the drag coefficient of an infinite wing with sweepback is given by

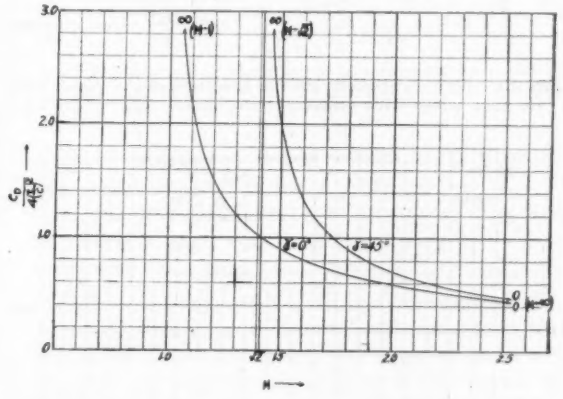


FIG. 8. Wave drag coefficients of two infinite wings (zero and 45° sweepback).

$$C_D = C_{D_0}\cos\gamma\,\frac{\sqrt{M^2-1}}{\sqrt{M^2\cos^2\gamma-1}} = \frac{C_{D_0}}{\sqrt{1-\beta^2}} \qquad (5.1)$$

where C_{D_0} is the wave drag coefficient of a normal wing with the same section and same chord measured in flight direction.

For $\beta > 1$, $C_D = 0$. Fig. 8 shows the drag coefficient for two wings with infinite span: one without sweepback ($\gamma = 0$) and one with 45° sweepback. It is seen that the occurrence of wave drag is postponed from $M = 1$ to $M = \sqrt{2} = 1.414$. However, for $M > \sqrt{2}$, the drag of the wing with sweepback is always larger than that of the normal wing.

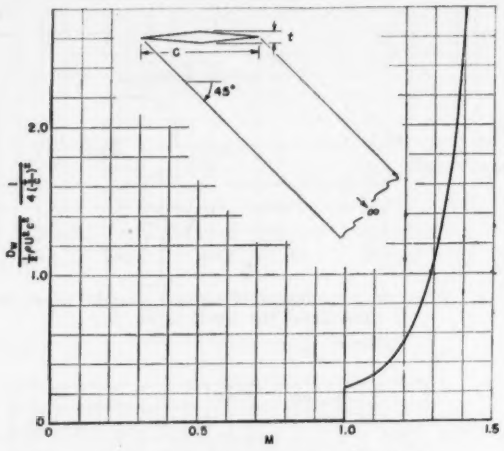


FIG. 9. Drag coefficient of a half infinite wing with sweepback (referred to an area equal to the square of the chord).

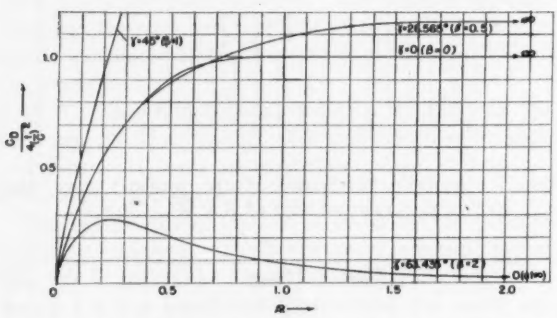


FIG. 10. Wave drag coefficients as functions of the aspect ratio for four wings with different sweepbacks. ($M = 1.414$.)

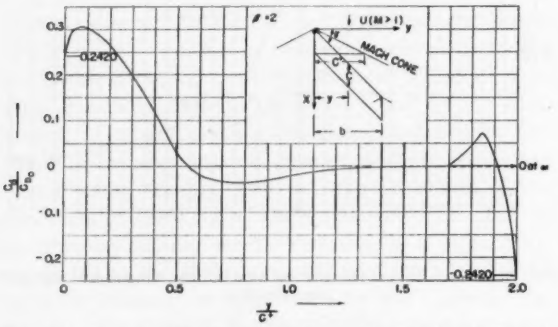


FIG. 11. Spanwise distribution of the sectional wave drag coefficient for wings with sweepback.

(c) *Half Infinite Wing with Sweepback and Wing with Finite Aspect Ratio*

The equation for the wave drag of a half infinite uniform wing with double wedge section and sweepback in the case of $\tan \gamma > \sqrt{M^2 - 1}$ is the following:

$$D = \rho U^2 c^2 \left(\frac{t}{c}\right)^2 \frac{2 \log 2}{\pi} \frac{\sin \gamma \cos^2 \gamma}{[1 - M^2 \cos^2 \gamma]^{1/2}} \qquad (5.2)$$

In this equation γ is the sweepback angle, c is the chord measured in flight direction, and t is the thickness of the section. It is seen that the wing has finite wave drag, although the span is infinite. The drag coefficient, referred to an area equal to c^2 as function of Mach Number, is shown in Fig. 9 for $\gamma = 45°$. When $M > \sqrt{2}$, the drag is, of course, infinite.

In the case of sweepback and finite aspect ratio, the drag coefficient will be a function of the two parameters A.R. $\sqrt{M^2 - 1}$ and β. For example, for the double wedge wing

$$C_D = \frac{4(t/c)^2}{\sqrt{M^2 - 1}} \varphi(\text{A.R.}\sqrt{M^2 - 1}, \beta)$$

The parameter A.R. $\sqrt{M^2 - 1}$ is the ratio of the span to the length $c^* = c \tan \alpha$ (α = Mach angle). The geometrical meaning of c^* is analogous to that defined before: the radius of the base of the tip Mach cone measured from the tip of the trailing edge.

The behavior of the drag coefficient is different for $\beta > 1$ and $\beta < 1$—i.e., according to whether the velocity component normal to the wing axis is subsonic or supersonic. In Fig. 10 the drag coefficients of three wings with different sweepback are compared with that of a normal wing. The Mach Number chosen for this comparison is equal to $M = \sqrt{2}$; the three sweepback angles are equal to 26.6°, 45°, and 63.4°. If the sweepback is equal to 45°, the velocity component normal to the wing axis is equal to sound velocity. The drag coefficients are represented versus geometrical aspect ratio. It is seen that the behavior of the drag curve for 26.6° sweepback ($\beta = 1/2$) is similar to that found for the normal wing, with the exception of a larger asymptotic value for the two-dimensional case and an extension of the tip effect to larger aspect ratios. The wing with 45° sweepback ($\beta = 1$) is, of course, unfavorable for this Mach Number. The drag coefficient increases to infinity for the two-dimensional case. A wing with 63.4° sweepback ($\beta = 2$) is extremely favorable as to drag in comparison with the normal wing. It is seen, for example, that, for A.R. = 1, $C_D/C_{D_0} = 0.092$—i.e., the sweptback wing has only 9 per cent of the drag of a normal wing of the same geometrical aspect ratio. It was mentioned that the theoretical wave drag coefficient of a normal double-edge wing, at the Mach Number considered, is equal to 0.0144; the wave drag coefficient of the same wing with 63.4° sweepback would be only 0.00132. For 26.6 and 45° sweepback, $C_D/C_{D_0} = 1.15$ and 2.38, respectively; hence, the corresponding drag coefficients would be equal to 0.0165 and 0.0343.

Fig. 11 is instructive as to the mechanism of the drag in the case of subsonic normal velocity component. The diagram represents the distribution of the section drag coefficient compared with the drag coefficient of a two-dimensional normal wing at the same Mach Number. The abscissa is the ratio y/c^*. The calculation was carried out for A.R. $\sqrt{M^2 - 1} = b/c^* = 2$ and $\beta = 2$. It is seen that the effects of the upstream

and downstream wing tips are clearly separated. The effect of the upstream tip is significant over a portion of the span slightly greater than c^*. The amount of drag corresponding to this effect is about equal to the drag of the half infinite wing with the same sweepback. The effect of the downstream wing tip extends to a relatively small portion of the span equal to $c^*/(1+\beta)$. The drag resulting from this tip effect is zero.

These considerations show a relatively easy way to calculate with good approximation the drag of a swept-back wing with arbitrary aspect ratio, provided A.R. $\times \sqrt{M^2-1} > 1$. Since in this case the total drag is about the same as the tip drag of a half infinite drag, one obtains from Eq. (5.2)

$$C_D = 4\left(\frac{t}{c}\right)^2 \frac{1}{\sqrt{M^2-1}} \frac{\log 2}{\pi} \frac{\beta}{(\beta^2-1)^{3/2}\,\text{A.R.}} \qquad (5.2a)$$

This approximate value is shown by the dotted line in Fig. 12 for the value of the sweepback parameter $\beta = 2$. The full line gives the exact value of the ratio C_D/C_{D_0} for the same case. The approximation furnished by Eq. (5.2a) is satisfactory almost down to A.R.$\sqrt{M^2-1} \cong 0.3$. If the aspect ratio is further reduced, C_D/C_{D_0} reaches a peak and after that decreases to zero. It is evident that, in the immediate neighborhood of the point A.R.$\sqrt{M^2-1} = 0$, C_D/C_{D_0} becomes independent of β—i.e., wings of any aspect ratio behave in the same way.

The following consideration is of interest: Let two identical wings with the same sweepback be located on a common axis. If the two wings have sufficiently large aspect ratio and are sufficiently far apart, each one has its own wave drag of equal amount. If the two wings approach each other, there is an interference that reduces the drag of the rear wing. When the two wings are matched together tip to tip, the drag of the rear wing entirely disappears. A somewhat analogous interference effect is well known in the theory of the induced drag in subsonic flow.

(d) *Arrowhead Wings*

The equation for the wave drag of an infinite arrowhead wing in the case of $\tan\gamma > \sqrt{M^2-1}$ is the following:

$$D = \rho U^2 c^2 \times \left(\frac{t}{c}\right)^2 \left(\frac{2\log 2}{\pi}\right)\left[\frac{(1+2\sin^2\gamma - M^2\cos^2\gamma)\cos^2\gamma}{\sin\gamma\,(1-M^2\cos^2\gamma)^{3/2}}\right] \qquad (5.3)$$

Fig. 13 shows the value of $C_D/[4(t/c)^2]$ as a function of Mach Number for an arrowhead wing with 45° sweepback and aspect ratio A.R. = 8. It is seen that the drag coefficient has a finite peak for $M = \sqrt{2}$—i.e., at the Mach Number that corresponds to $\beta = 1$. For $\beta < 1$—i.e., when the sweepback is not sufficiently

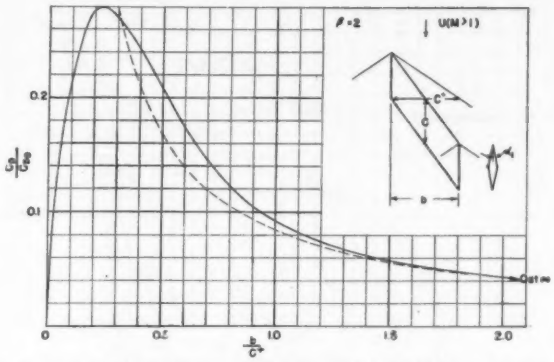


FIG. 12. Wave drag coefficient as a function of the aspect ratio for wings with sweepback ($\beta = 2$). Dotted line corresponds to approximation by Eq. (5.2a).

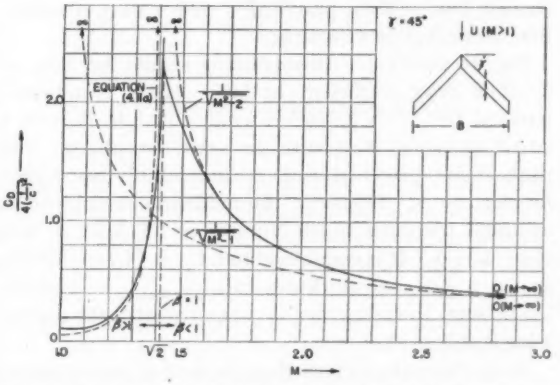


FIG. 13. Wave drag coefficient of an arrowhead wing with large aspect ratio (A.R. = 8) as a function of Mach Number.

large to cause subsonic flow normal to the wing axis—the conditions can be described as follows: For $M > 1.61$, the drag coefficient of the arrowhead wing is equal to that of a single two-dimensional wing with the same span and sweepback. In this case $C_D = [4(t/c)^2] \div \sqrt{M^2-2}$ [compare Eq. (5.1)]. Between $M = 1.5$ and 1.61, this approximation is still satisfactory. Between $M = 1.41$ and 1.5, the approximate formula leads to too large values, tending to infinity for $M = \sqrt{2}$.

For $\beta > 1$—i.e., $M < \sqrt{2}$—one obtains a fair approximation (with the exception of the immediate neighborhood of $M = 1$ and $M = \sqrt{2}$) if one divides the drag given by Eq. (5.3) for the arrowhead wing with infinite aspect ratio by the actual wing surface. This leads, with $\gamma = 45°$, to the approximate formula

$$C_D = 4\left(\frac{t}{c}\right)^2\left[\frac{\log 2\,(4-M^2)}{\pi\,\text{A.R.}\,(2-M^2)^{3/2}}\right] \qquad (5.3a)$$

The values corresponding to Eq. (5.3a) are shown in Fig. 13 by a dotted line. For comparison, the values of $C_D/[4(t/c)^2]$ for a normal wing with infinite aspect ratio are also shown. The minimum value of the drag occurs at about $M = 1.08$. For 6 per cent thickness ratio,

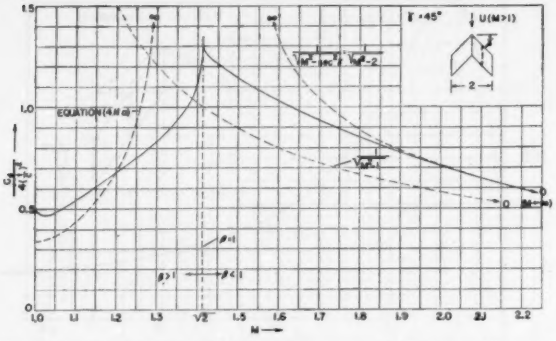


FIG. 14. Wave drag coefficient of an arrowhead wing with small aspect ratio (A.R. = 2) as a function of Mach Number.

$C_{D_{min.}} = 0.00082$. It is seen that this value is extremely low, so that, practically, such a wing combination has negligible wave drag.

Fig. 14 shows the corresponding results for A.R. = 2. The drag coefficient at large Mach Numbers—namely, for $M > 2.236$—is again equal to that of a two-dimensional wing with the same sweepback. The peak at $M = \sqrt{2}$ ($\beta = 1$) is considerably lower than for A.R. = 8. However, the minimum, which occurs at about the same Mach Number as for A.R. = 8, is much larger. It is equal to about $C_D = [4(t/c)^2](0.47)$, or for 6 per cent thickness ratio, $C_{D_{min.}} = 0.00678$. To be sure, the reduction as compared with the normal wing is still great.

It is interesting to note that the drag of an arrowhead wing remains unchanged when the flight direction is reversed. However, the spanwise drag distribution is different. For example, in the case of sweptback wings of sufficiently large aspect ratio, the total drag acts on the center section, whereas in the case of sweep forward, a considerable portion of the drag acts on the tips.

The invariance of drag with respect to reversal of flight direction is a special case of a general result of the linearized wave drag theory. The wave drag is independent of flight direction in any case in which the source distribution representing the flow is the same. Since, within the approximation used in this theory, the source distribution is inverted but not changed when the body is turned around relative to the flight direction, the theorem that the drag is independent of flow direction applies to a body of arbitrary shape. The body can be a flat one, like an airplane wing, or it can be a body of revolution. However, one should keep in mind that this is only true within the limits of validity of the linearized theory with approximate boundary conditions.

(6) The Mechanism of Lift

The theory of lift of an airplane wing moving at subsonic velocities is based on the concept of circulation. The origin of the circulation can be described in the following way. Consider an airplane wing initially at rest and suddenly given a forward velocity. The equations of motion admit a solution that represents a flow without circulation and therefore without lift. However, this flow has infinite velocity around the sharp trailing edge of the airfoil section. Since some viscosity always exists, the flow separates with consequent formation of a vortex, called the "starting vortex." The reaction to the starting vortex produces circulation around the airfoil; the final magnitude of the circulation is determined by the condition of smooth flow at the trailing edge, called the Kutta-Joukowski condition. The condition of smooth flow is equivalent to the statement that the lift per unit area is zero at the trailing edge; the pressures acting on the edge at the upper and lower surface of the airfoil are equal.

It is easily seen that this process is, in general, impossible in the case of supersonic motion. Consider, for example, the two-dimensional case—i.e., the case of an airplane wing of infinite span—the trailing edge being normal to the flow direction. It is evident that, according to the rule of forbidden signals, no process at the trailing edge can have an effect upstream. Correspondingly, the lift density at the trailing edge may well have a finite value. This also follows from Ackeret's rule that the pressure produced at every element of the surface depends on the local angle of attack. Therefore, for example in the case of a flat plate with an angle of incidence i, one obtains a uniform pressure of the amount $(2i/\sqrt{M^2 - 1})$ $(\rho U^2/2)$ acting on the lower surface and a uniform suction of the same magnitude on the upper surface (Fig. 15). This causes a discontinuity of pressure at the trailing edge, the effect of which, however, is restricted to the flow downstream. Hence, a flow around the airfoil which would equalize the pressures at the trailing edge cannot be created.

It is seen that the lift distribution near the leading edge is also different in both cases. In the subsonic case, the theory gives infinite lift density for a plate with mathematically sharp leading edge. This is consistent with the fact that the air is supposed to flow around the leading edge with infinite velocity. One might object to the inconsistency that the theory admits infinite velocity at the leading edge and forbids

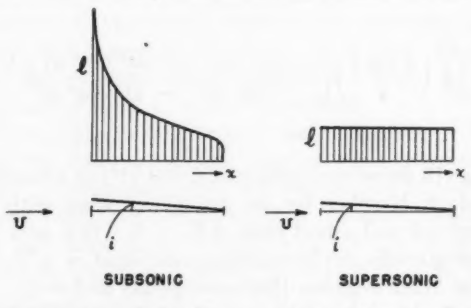


FIG. 15. Lift distribution over the chord of a flat plate.

the same at the trailing edge. The reason is that, although the fluid separates at the sharp leading edge, it will again become attached to the upper surface, at least below the stalling angle. In the case of suitably rounded leading edge, the flow will follow the surface continuously. The flow around the edge produces negative pressure, which in the simplified theory of the flat plate or infinitely thin wing is taken into account by the assumption of a concentrated force at the nose. This force balances the horizontal component of the resulting pressure forces on the plate and makes the drag of a two-dimensional wing in a perfect fluid equal to zero, as required by d'Alembert's theorem. In the supersonic case—more exactly, if the flow velocity normal to the leading edge is supersonic—no such flow occurs around the leading edge. The pressure difference between upper and lower surfaces will be finite, and we do not have the benefit of a suction force acting at the leading edge. The resultant of the pressure forces is normal to the plate, and its horizontal component represents an actual drag.

The theory of lift of a wing with finite span moving at subsonic speed uses particular solutions of the linearized flow equations which represent elementary horseshoe vortices. A horseshoe vortex consists of the so-called bound vortex, which carries the lifting force, and two free vortices. These latter create the induced velocities (Fig. 16). It is known that the kinetic energy of the two free vortices, which remains in the air behind the moving wing, represents the work done against the induced drag—i.e., the work necessary to create lift.

In the supersonic case it is possible to build up a flow by means of analogous particular solutions of the wave equation. Each such solution represents a concentrated lift force. The elementary flow pattern in this case is

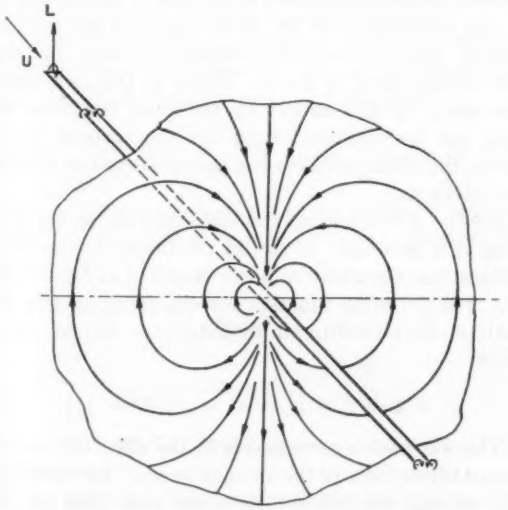


FIG. 16. Horseshoe vortex in subsonic flow.

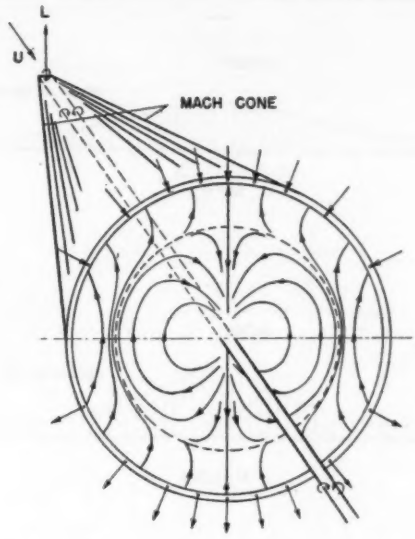


FIG. 17. Horseshoe vortex in supersonic flow.

restricted to the surface and interior of the Mach cone whose vertex lies at the point of application of the lift force (Fig. 17). The lines indicating the direction of flow in a plane normal to the main flow direction are also shown in the figure. It is seen that, at a large distance behind the wing, the flow in the neighborhood of the axis is identical to that produced by a subsonic horseshoe vortex. One concludes from this consideration that the induced drag exists in the supersonic case as well. Since the wave drag is also due to induced velocities, in the supersonic case, the specific expression "induced vortex drag" appears more appropriate. One finds, in fact, that the relation between induced vortex drag and lift distribution is the same for subsonic and supersonic flow, at least within the approximation corresponding to the linearized theory.

However, in the supersonic case the production of lift requires, in addition to the induced drag, a certain amount of wave drag corresponding to the energy radiated to infinity along the Mach cone. This drag is found to be proportional to the square of the lift produced—i.e., it follows a similar law as the induced drag. The drag acting on the flat plate, which was discussed previously, is evidently a simple example of the wave drag produced by lift.

It is interesting to compare the pressure produced by a wing moving at subsonic and supersonic velocity over the ground (Fig. 18). Consider first, for the sake of simplicity, the wing of infinite span. In the case of $M \to 0$—i.e., at a velocity that is small in comparison with sonic velocity—the weight of an airplane wing flying over the ground is carried by pressures created on the ground and distributed over a large area. The center of the pressures lies vertically under the center of gravity of the airplane, so that the weight of the air-

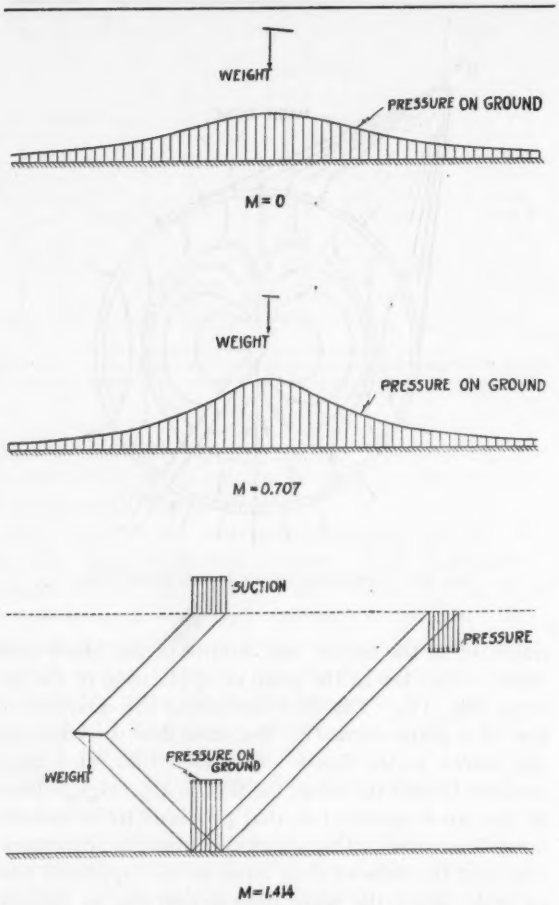


FIG. 18. Pressure distribution on the ground produced by an airfoil with infinite span.

plane and the resultant of the pressures acting on the ground are in equilibrium. For increasing Mach Numbers, one finds that the area over which the bulk of the pressure is distributed becomes narrower and narrower. In the supersonic case, the pressure effect below the wing is restricted to a strip inclined at the Mach angle and intersecting the ground at a certain distance behind the plane. If the plane flies at high altitude, this distance is evidently extremely large. It is seen that in this case the equilibrium of the forces is restored by the additional pressure effects shown in Fig. 18. In fact, if the weight of the plane per unit span is denoted by W, the resultant of the pressures acting on the ground is equal to W and, together with the weight, constitutes a couple. Consider now a horizontal control plane above the airplane. It is seen that a total negative pressure force equal to $\frac{1}{2}W$ is acting on the intersection of the horizontal plane and the wavestrip extending from the wing upwards, and an equal amount of positive pressure force is acting on the intersection of the plane and the wavestrip reflected from the ground. It is easily seen that the level arms of the two couples are in the ratio 1 to 2, so that the whole system is in equilibrium.

Similar considerations can readily be made for the case of wings with finite span (Fig. 19).

(7) THE LINEARIZED THEORY OF LIFTING SURFACES

(*a*) *Two-Dimensional Case*

From the fundamental fact that the supersonic flow of the velocity U produces a normal pressure equal to

$$(2\delta/\sqrt{M^2 - 1})\ (\rho U^2/2)$$

where δ denotes the local angle of inclination of the surface, one easily obtains the equations for the lift and drag coefficients of an arbitrary thin section as follows:

$$C_L = 2(\bar{\delta}_u + \bar{\delta}_l)/\sqrt{M^2 - 1} \tag{7.1}$$

$$C_D = 4\bar{\delta}^2/\sqrt{M^2 - 1} \tag{7.2}$$

In Eq. (7.1), δ_u and δ_l denote the angles of inclination at an arbitrary point of the upper and lower surfaces, respectively. For a symmetric section with the angle of incidence i, Eqs. (7.1) and (7.2) yield the results:

$$C_L = 4i/\sqrt{M^2 - 1} \tag{7.3}$$

$$C_D = (4/\sqrt{M^2 - 1})\ (\bar{i}^2 + \bar{\delta}^2) \tag{7.4}$$

For a double wedge section of thickness ratio t/c, one obtains

$$C_D = (4/\sqrt{M^2 - 1})\ [\bar{i}^2 + (t/c)^2]$$

It is seen that, if the drag due to friction and flow separation is neglected, the maximum of the lift drag ratio occurs when $i = t/c$, and its value is equal to half the reciprocal of the thickness ratio.

(*b*) *Lifting Surface with Finite Span*

Both methods presented in Section 4 can be applied to the calculation of the flow produced by a lifting surface, provided the lift distribution over the plan form of the wing is given. Since, in the approximation used, the flow caused by the finite thickness of a wing can be separated from the flow caused by lift forces, the lifting surfaces can be considered to be without thickness.

Instead of the sources and sinks used in the theory of drag, it is necessary to use, in the theory of the lifting surface, the elemental solution sketched in Figs. 16 and 17. The potential function corresponding to this flow pattern can be written in cylindrical coordinates (x, r, and θ)

$$\varphi = (\cos\theta/r)\ [x/\sqrt{x^2 - r^2(M^2 - 1)}] \tag{7.5}$$

This expression corresponds to the effect of a concentrated lifting force of the amount $2\pi\rho U$. By computing the vertical velocity produced by such flow patterns continuously distributed over the plan form of the

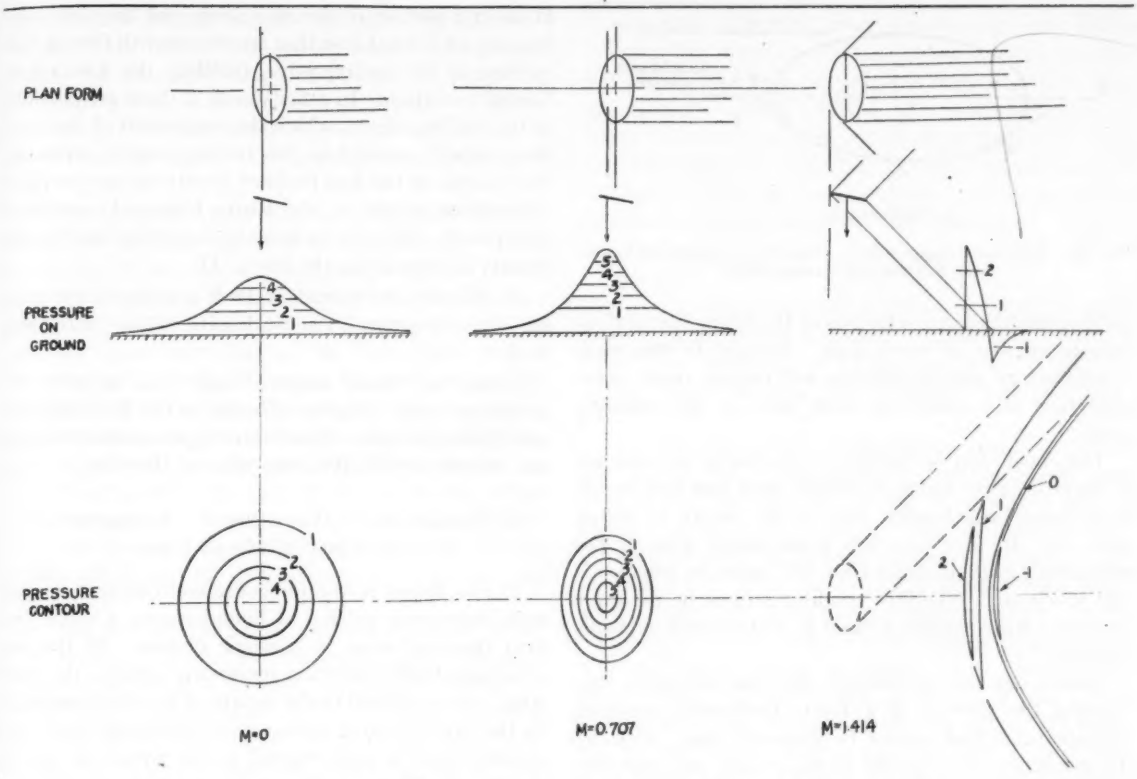


FIG. 19. Pressure distribution on the ground produced by an airfoil with finite aspect ratio.

wing, it becomes possible to compute the camber distribution that would produce the given lift distribution. Also, since the flow is completely known, the wing drag can be computed. Finally, the induced drag is calculated in the classical way and the problem is so far solved.

The method of the Fourier integrals will greatly simplify these calculations. The results analogous to those presented in Section 4 are the following:

(1) One starts with the Fourier analysis of the lift distribution and expresses the lift per unit area distributed over an arbitrary section in the form

$$l = 2\rho U^2 \frac{a}{\sqrt{M^2 - 1}} \int_0^\infty d\nu (g_1 \cos \nu t - g_2 \sin \nu t) \quad (7.6)$$

(2) Then the contribution of two arbitrary sections S and S^* to the total wave drag of the wing is given by the equation:

$$D = \frac{\pi \rho U^2 a^2}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dy\, dy^* \int_0^\infty d\nu (g_1 g^*_1 + g_2 g^*_2) \frac{J_1(\nu |y - y^*|)}{|y - y^*|} \quad (7.7)$$

where J_1 denotes the Bessel function of the first kind and the order one.

(3) In this case, also, the acoustic analogy can be easily constructed. A lift distributed over a section corresponds to a pulse transmitting vertical impulses to the air over a limited period of time. Every section represents an oscillator. If we imagine that the impulse is transmitted by the motion of a physically existing surface, this, of course, has to disappear after the pulse is terminated.

It can easily be seen that in this case the energy transmitted to the fluid consists of two parts. A certain amount of energy will be radiated to infinity; another part remains in the fluid in the domain in which the impulse has been transmitted. The first amount of energy corresponds to the wave drag; the second amount, to the induced vortex drag.

(4) The wing camber distribution that produces a given lift distribution can be calculated by the Fourier method without intrinsic difficulty.

However, the direct problem, which in many cases may be more interesting to the designer—namely, the determination of the lift distribution for a given wing shape (i.e., given plan form, camber, and angle of attack)—is much more difficult. The exact solution of the analogous problem causes great difficulties in the subsonic case as well. However, in the subsonic case the concept of the lifting line gives a fair approximation, at least for wings with large aspect ratio and without large sweepback. Unfortunately, the concept of lifting line cannot be used in the supersonic case, since it

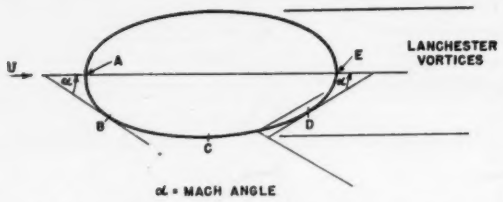


FIG. 20. Schematic diagram for a wing with subsonic and supersonic leading and trailing edges.

yields infinitely large velocities at the lifting line and an infinite amount of wave drag. Because of this fact, a satisfactory general solution will require much more analytical and numerical work than in the subsonic case.

The conclusion of Section 5 concerning the absence of wave drag for wings of infinite span and sufficiently large sweepback applies also to the theory of lifting surfaces. In fact, one sees immediately that, if the sweepback angle is larger than 90° minus α, where α is the Mach angle, the flow conditions must be the same as over a wing moving normal to its axis with subsonic velocity.

Hence, one has to conclude that the statement concerning the absence of a Kutta-Joukowski condition in supersonic flow cannot be generally true. One has to introduce the concept of supersonic and subsonic trailing edges.

For example, it is evident that in the case of the wing with large sweepback, signals emitted from the trailing edge may cover a certain part of the wing plan form and reach even a portion of the leading edge. Consequently, the statement that trailing edge conditions cannot influence the flow over the wing is evidently not correct in such a case.

The fundamental ideas entering the problem can best be illustrated by considering a wing with elliptic plan form (Fig. 20). The tangents to the ellipse parallel to the flow direction and to the direction of the Mach lines subdivide the boundary of the wing plan form into four types:

(1) The leading edge from A to B will be called a supersonic leading edge. The local flow conditions are the same as at the leading edge of a supersonic normal wing with infinite span. One has at the leading edge a finite lift per unit surface, determined by the normal component of the flow velocity and the local angle of attack.

(2) The boundary B to C will be called a subsonic leading edge. As in the case of a sharp-edged flat plate in subsonic flow, the lift density at the leading edge is infinite but integrable. If an infinite lift density exists, a horizontal suction force will also appear at the leading edge.

(3) The portion of the boundary between C and D has the character of a trailing edge in subsonic flow. It is seen that signals emitted from points between C and D cover a portion of the plan form, and they can produce an additional flow that secures smooth flow at this portion of the trailing edge, fulfilling the Kutta-Joukowski condition. In other words, if there are portions of the trailing edge at which the component of the main flow velocity normal to the trailing edge is subsonic, the solution of the flow problem is only unique provided a condition similar to the Kutta-Joukowski condition is imposed. This can be done by requiring that the lift density be zero along the line C–D.

(4) Finally, the boundary D–E is a supersonic trailing edge. In general, the lift density will be finite along D–E.

Along the whole trailing edge—i.e., between the points at which tangents parallel to the flow direction touch the plan form—Lanchester-type vortices originate and extend to infinity in the wake of the wing.

(8) Escape from Wave Drag. Interference, Sweepback, Delta Wing

It was shown in the foregoing sections that, in general, supersonic motion of bodies causes a wave drag that does not exist in subsonic motion. In the case of slender bodies and thin symmetric airfoils, the wave drag is proportional to the square of the thickness ratio; in the case of lifting surfaces, an additional wave drag occurs which is proportional to the square of the lift produced. These results give a general hint in favor of small thickness ratios and small lift coefficients. However, this general advice is limited by requirements of weight and strength of construction. Therefore, the airplane designer will welcome ideas leading to reduction of the wave drag without going to extreme slenderness and small lift coefficients.

The physical significance of wave drag consists of a continuous transport of momentum from the moving body to the air to infinity if there are no boundaries. One can also say that energy is continuously transmitted to the air, radiated from the moving body to infinity. Work done by a propulsive thrust is necessary to furnish this energy. Therefore, if there is a method to prevent the energy from leaving the moving system, there will be no wave drag, and propulsive thrust will be spared. This offers the possibility, at least theoretically, of escaping totally or partially from wave drag by appropriate interference between components of the moving system. It is possible, for example, to combine two airfoil sections with flat external surfaces parallel to the flow direction, so that their internal curved surfaces face each other. By properly designing the two inner surfaces, one can prevent the reflection of pressure waves on the inner surfaces, and the air leaves the system without change in velocity or energy. Of course, this system does not produce lift.

It is also possible to design a biplane—as was first indicated by A. Busemann—in such a way that the expansion waves emanating from the upper surface are

compensated by the pressure produced at the lower surface of the other. This measure may substantially reduce the total wave drag. A biplane, of course, has disadvantages from several points of view. Nevertheless, the problem of interference action should be studied further with a view toward reduction of wave drag.

Another general idea has already been mentioned in connection with the supersonic airfoil theory. It was shown that, in the case of a sweptback wing with infinite span, the wave drag disappears when the sweepback is sufficiently large for the flow velocity normal to the wing axis to become subsonic. We have shown that in the case of wings with finite span, the wave drag is also essentially reduced by sufficiently large sweepback. Of course, when the velocity normal to the leading edge approaches sonic velocity, one will have a similar increase in drag, as in the case of straight normal wings in the transonic speed range, due to shock and separation. Furthermore, the sweptback wing design has its peculiar problems and difficulties because of the plan form.

One serious problem is the appropriate choice of airfoil sections. If a wing has a subsonic leading edge, a supersonic section will produce large peaks of negative pressure at the leading edge, eventually causing separation. Therefore, the designer, in this case, should use sections suitable for transonic flow, similar to the laminar flow sections used in airplanes designed for high subsonic speeds. At the center section of an arrowhead wing, or at the sections of a sweptback wing near the fuselage, supersonic conditions prevail, and one must therefore probably change the basic shape of the section along the span.

The aerodynamic design of control surfaces constitutes a branch of supersonic aerodynamics which is very much in need of fundamental investigation. The suggestion of boundary-layer control as a direct method of airplane control by changing lift distribution over the wing deserves careful study. It may have extraordinary possibilities and help to eliminate the difficulties connected with the change from subsonic to supersonic flow régimes.

One variation of the sweepback scheme is the wing with triangular plan form, the so-called delta wing. This plan form also has considerable theoretical interest, since, for delta wings with certain simple angle of attack distributions, the direct problem of the wing theory can be solved in a relatively easy way. In the foregoing sections two general methods of computing flow produced by slender bodies and airfoils were presented: the methods of sources and of Fourier integrals. A third elegant method is that of the so-called "conical flows." A flow is called conical if its three velocity components are constant along straight lines that diverge from one point. The simplest example is a flow along a cone of circular cross section. Conical flow is a generalization of this flow pattern. It is convenient to introduce spherical polar coordinates: the radius vector r, the meridian angle φ, and an azimuth parameter ξ (connected with the azimuth angle ω by the relation $\cosh \xi = \cot \omega/\sqrt{M^2 - 1}$). Since the three velocity components u, v, and w of a conical flow are independent of r, φ and ξ are chosen as independent coordinates. Then it can be shown that u, v, and w satisfy the Laplace equation; for example,

$$(\partial^2 u/\partial \xi^2) + (\partial^2 u/\partial \theta^2) = 0$$

Hence, the mathematical problem of finding conical flows to satisfy given boundary conditions is relatively easy. For example, the elegant methods of the conformal transformation are available for this purpose.

The flow produced by a thin delta wing is a conical flow within the approximation of the linearized wing theory, provided the local angle of attack is only a function of the azimuth angle. The simplest example of this kind is the delta wing with constant local angle of attack—i.e., a flat plate with triangular plan form. In this case, closed mathematical expressions can be obtained for the lift and drag distribution of an infinite triangular wing. Then, according to the rule of forbidden signals and the general properties of plan forms with subsonic and supersonic trailing edges, the solution for the infinite wing is valid without change also for a finite wing, provided it has supersonic trailing edges.

Consider, for example, a triangular wing with sufficiently large sweepback so that the leading edge is subsonic. The trailing edge is assumed to be normal to the flight direction. A simple analysis shows that the lift density is infinite but integrable along the leading edges and finite and different from zero at the trailing edge, as follows from general considerations previously given. It is found that the spanwise lift distribution is elliptic. The lift per unit span is equal to

$$(\rho U^2/2)\ (is/E)[\sqrt{1 - (4r^2/s^2)}]$$

The total lift is equal to

$$(\rho U^2/2) i s^2 (\pi/2E)$$

In these equations E is a numerical factor given by the elliptic integral of the second kind:

$$E = \int_0^{\pi/2} \sqrt{1 - [1 - (\tan^2 \omega_0/\tan^2 \alpha)] \sin^2 \beta}\, d\beta$$

The total drag is equal to

$$D = Li - (Li/2E) \sqrt{1 - (\tan^2 \omega_0/\tan^2 \alpha)} \quad (8.1)$$

where α is the Mach angle and i is the angle of incidence. The quantities r, s, and ω_0 are defined in Fig. 21.

The first term represents the sum of the horizontal components of the pressures acting on the upper and lower surfaces. The second, negative, term corresponds to the suction acting at the leading edge. One can also easily separate the wave drag from the induced

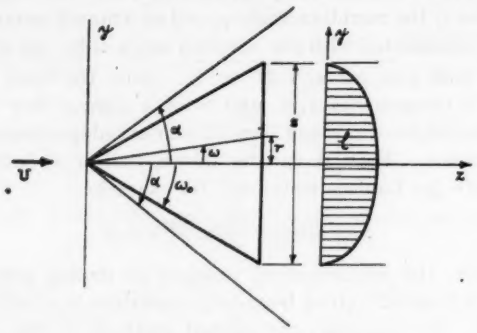


FIG. 21. Delta wing.

drag. The amount of the induced drag is given by the simple relation

$$D_i = Li/2E \tag{8.2}$$

The rest, represented by the equation

$$D_w = Li\,\{1 - [1 + \sqrt{1-(\tan^2\omega_0/\tan^2\alpha)}]/2E\} \tag{8.3}$$

gives the wave drag.

One can show by means of calculation that, with sufficiently large sweepback, favorable values of lift/drag ratios can be reached with satisfactory lift coefficients.

The drag produced by thickness of the wing is not included in this calculation. The final answer on the practical merit of the delta wing will be given by further comprehensive theoretical and experimental investigations.

The method of conical flows, which led to such simple results in the case of the triangular wing, is also useful for the solution of the lift problem for a large class of wings with various sweepback and taper combinations. It can also be used in the theory of drag of wings with given sectional shape, especially if the section is composed of straight lines.

(9) Friction and Boundary Layer

In the foregoing sections two assumptions were made: absence of viscosity and small magnitude of the perturbations caused by the presence of a body in a supersonic stream. This and the following sections contain a few remarks on the influence of viscosity and finite perturbation.

There is not much information available on the magnitude of the friction between a solid and air moving at supersonic speed relative to the solid surface. However, pressure drop measurements in pipes indicate that the friction coefficient obtained for subsonic flow can be applied to the supersonic case as well, at least in the case of turbulent flow. Ballistic experience confirms this result. It appears that the frictional resistance of projectiles is of the order of magnitude to be expected by extrapolation of the friction coefficients from subsonic to supersonic speeds as functions of the Reynolds Number only.

The boundary-layer theory was extended to the supersonic velocity range by various authors in the case of laminar flow. This is, of course, no longer solely a problem of aerodynamics; it is rather one of aerothermodynamics (a term introduced by G. A. Crocco), since the heat produced by viscosity and the heat exchange across the boundary layer are essential factors which cannot be neglected. One interesting result is that if the wall is thermally insulated, the temperature of the air immediately adjacent to the wall reaches the value corresponding to an adiabatic compression of the gas to stagnation pressure, although no pressure rise occurs. If the temperature of a conducting wall surface is lower than this value, heat is transferred to the wall. Hence, even if there is a considerable temperature difference between a moving hot body and cool outside air at a certain flight Mach Number, the cooling is reversed to heating. This is due to the heat produced by internal friction in the boundary layer. The reversal occurs at

$$M = \sqrt{[2/(\gamma - 1)][(T_w - T_s)/T_s]}$$

where T_w and T_s are the wall and stream temperatures, respectively. When H. S. Tsien and the present author considered this problem in 1938, it appeared to have merely academic interest. However, it is evident that today it is a practical question, for example, with reference to V-2-type rockets. The most recent progress in the integration of the aerothermodynamic equations relative to laminar boundary layers in compressible fluids has been made by L. Crocco.

The question of stability of the laminar boundary layer in an incompressible fluid has been cleared up in final form by the mathematical work of C. C. Lin and the experimental research of H. L. Dryden. The stability problem for laminar compressible boundary layer was recently investigated by C. C. Lin and L. Lees. In general, if there is a heat flow through the wall, the compressibility appears to have a stabilizing effect, whereas, if the wall is thermally insulated, the effect is in the opposite direction. However, one must bear in mind that, as will be pointed out in Section 12, the boundary-layer theory applies only to the case where the outside flow determines the boundary layer completely, and no interaction from the boundary layer to the outside flow can be expected. Hence, the problem of stability in a broader sense should include the mutual interaction between boundary layer and outside flow, especially between boundary layer and shock wave. At high altitude—i.e., in a medium of low density—the heat radiated must also be taken into account. The cooling of the wall by radiation can increase the stability of the laminar boundary layer to a large extent.

As to the theory of the fully developed turbulent boundary layer and turbulent separation, these problems have not even been solved for the case of incompressible fluids. The problem of separation of supersonic streams is closely connected with the problem of formation of shock waves. This question will be discussed in Section 12. It has a fundamental bearing on the problem of transonic flow.

(10) The Exact Theory of Supersonic Flow

If one wants to proceed to a more exact analysis of the forces acting on bodies in a supersonic stream beyond the linear approximation based on the assumption of small perturbations, the simple rules of Section 2 and the theories of drag and lift presented in Sections 3 to 7 must be modified in various essential aspects.

(a) In the linear approximation, the velocity of sound is assumed to be constant throughout the whole field of the fluid motion, and equal to the velocity of sound in the undisturbed flow. Now it is known that the velocity of sound is proportional to the square root of the temperature of the gas; therefore it will vary from point to point if the pressure change is not exactly isothermal. If the variation of the velocity of sound and the influence of the difference between local and main flow velocity on the wave propagation are taken into account, the equations of motion are no longer linear and the method of superposition of particular solutions is no longer available to the mathematician. Correspondingly, one can no longer talk of Mach cones traversing the whole flow. The concept of the Mach cone is restricted to infinitesimal local Mach cones, which determine the pressure propagation in the infinitesimal neighborhood of one point. The axis of such an infinitesimal Mach cone is parallel to the direction of the local velocity, and its vertex angle corresponds to the local Mach Number. The method of superposition of particular solutions valid for the whole field must be replaced by a method of step-by-step construction. In the case of two-dimensional flow and three-dimensional flow with axial symmetry, such step-by-step integrations have been carried out using numerical or graphical methods. The lines of intersection of the elementary Mach cones with the plane of the two-dimensional flow, or with a meridian plane of the axially symmetric flow, constitute networks of curves called the characteristic curves. The tangents of the flow lines are the bisectors of the characteristics, the angle between a flow line and each of the two characteristic curves intersecting it being equal to the local Mach angle.

The method of characteristics makes it possible to obtain essential refinements of the drag computations of the linearized theory. This is especially valuable for a check of the degree of approximation which can be attained by the latter.

(b) A second point that has to be emphasized is the fact that the velocity of sound is the velocity of propagation of infinitesimally small pressure variations, whereas a finite pressure rise propagates at an essentially higher speed. The velocity of propagation is a function of the magnitude of the pressure rise. Therefore if the pressure rise produced by the moving body has finite magnitude, the rule of forbidden signals as stated in Section 2 is not exactly true. In general, the surface that separates the zone of silence from the zone of action is no longer a straight cone starting from the source of perturbance; in most cases it is a curved surface at which pressure, density, and velocity undergo sudden finite changes. The sudden change of these characteristic quantities is commonly called a shock, and the surface at which the change occurs is called a shock wave. This terminology reminds us that the discontinuity is caused by wave propagation—namely, propagation of a wave front with finite amplitude at a speed that is greater than the velocity of sound.

The possibility of discontinuous change is not revealed by the linear theory, which deals only with infinitesimally small perturbations. It is indicated in certain cases by infinite values for velocity or pressure gradients.

When the velocity is perpendicular to the discontinuity surface, the shock is called a normal shock. In a normal shock only the magnitude of the velocity changes. The velocity upstream of the discontinuity surface must, of course, be supersonic, and it always becomes subsonic at the downstream surface. If the discontinuity surface is not normal to the velocity, the tangential component of the velocity remains unaltered by the transition through the surface. The velocity component normal to the surface changes from supersonic to subsonic magnitude. In the so-called hodograph diagram, the shock polar represents all the velocity vectors in which a given velocity vector, say the vector AB in the diagram of Fig. 22, can be discontinuously changed without violating the three pertinent dynamic and thermodynamic theorems—namely, those of the conservation of matter, momentum, and energy. The diagram shows that, if the initial velocity is given, the magnitude of the angle of deflection is limited. This

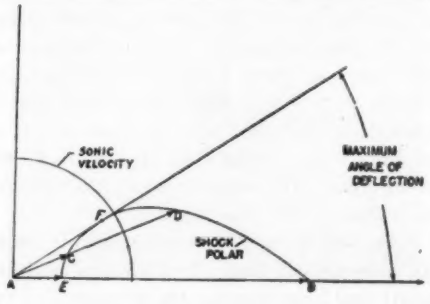


Fig. 22. Shock polar.

explains why, for example, in the case of a body with pointed nose, a shock wave can be attached to the conical nose if the vertex angle is not too large. For larger angles one obtains a so-called detached shock wave. The detached shock wave also appears if the body has a blunt nose or edge. Fig. 22 shows that in the case of attached shock waves there are theoretically two possibilities of sudden change of direction—namely, from the vector AB to AC or AD. For reasons that are theoretically not yet clear, at a pointed wedge or cone the change from AB to AD usually occurs—i.e., the change that involves smaller variation in the magnitude of the velocity. The angle of maximum deflection depends on the Mach Number and approaches zero when $M \rightarrow 1$; therefore, when the speed of a moving body passes through sonic velocity, a detached shock wave always occurs first. Theoretically, the detached shock appears at infinite distance ahead of the body when $M = 1$. Shadow photographs of a projectile passing through sonic velocity are shown in Figs. 23. With increasing Mach Number, the distance between nose and shock wave gradually decreases, as shown in Fig. 24. In this particular case (20° half vertex angle), the shock wave is first attached at $M = 1.18$.

Consider now a detached shock wave ahead of a wedge. Then point B in Fig. 22 corresponds to infinity, and E corresponds to the intersection of the shock wave with the plane of symmetry. As the Mach Number increases, the portion of the shock wave corresponding to the arc EF becomes smaller and smaller and vanishes when the shock wave becomes attached. This is in accordance with the fact mentioned above, that for higher Mach Numbers the portion of the shock wave in the neighborhood of the edge corresponds to the point D rather than to the point C.

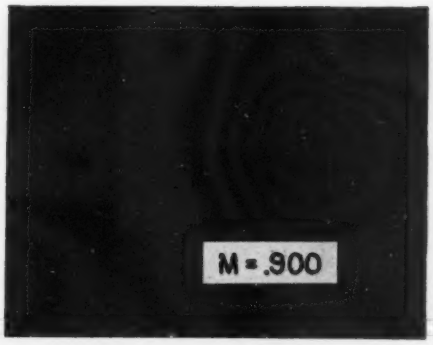
The process of sudden pressure rise is not reversible, since it involves an increase of the entropy of the gas. In other words, no sudden pressure drop can be maintained in a stationary flow. If one wants to express this fact in another way, one can say that sudden finite velocity decrease is possible, but sudden finite velocity increase is impossible, in a compressible fluid. This behavior of the compressible fluid in supersonic flow is illustrated by the flow pattern shown in Fig. 25. The fluid passing along the concave corner undergoes an instantaneous change in velocity and pressure, whereas the fluid passing around the convex corner changes its velocity and pressure continuously. The lines of constant pressure constitute a fan-like formation starting from the corner. The remarkable fact is that in the supersonic flow, the fluid may go around the corner without infinite velocity or separation of flow, whereas it is known that in the subsonic case either the velocity becomes infinite or the flow separates.

The discussion of the phenomena given in these lines refers to a nonviscous fluid. In a viscous fluid, because of viscosity and heat transfer, the pressure and velocity changes will always be continuous. However, it can be shown that the region in which the major part of the change takes place is of the order of magnitude of the mean free path of the gas molecules, and therefore, in general, very small, unless the density of the gas is extremely low. The thickness and the physical nature of the transition region are also influenced by the internal thermal properties of the gas—namely, by the distribution of thermal energy to the various degrees of freedom in a molecule. This effect is called the relaxation effect and is especially important in the case of certain gases with sluggish internal vibrations. The investigation of this last problem requires the methods of quantum mechanics.

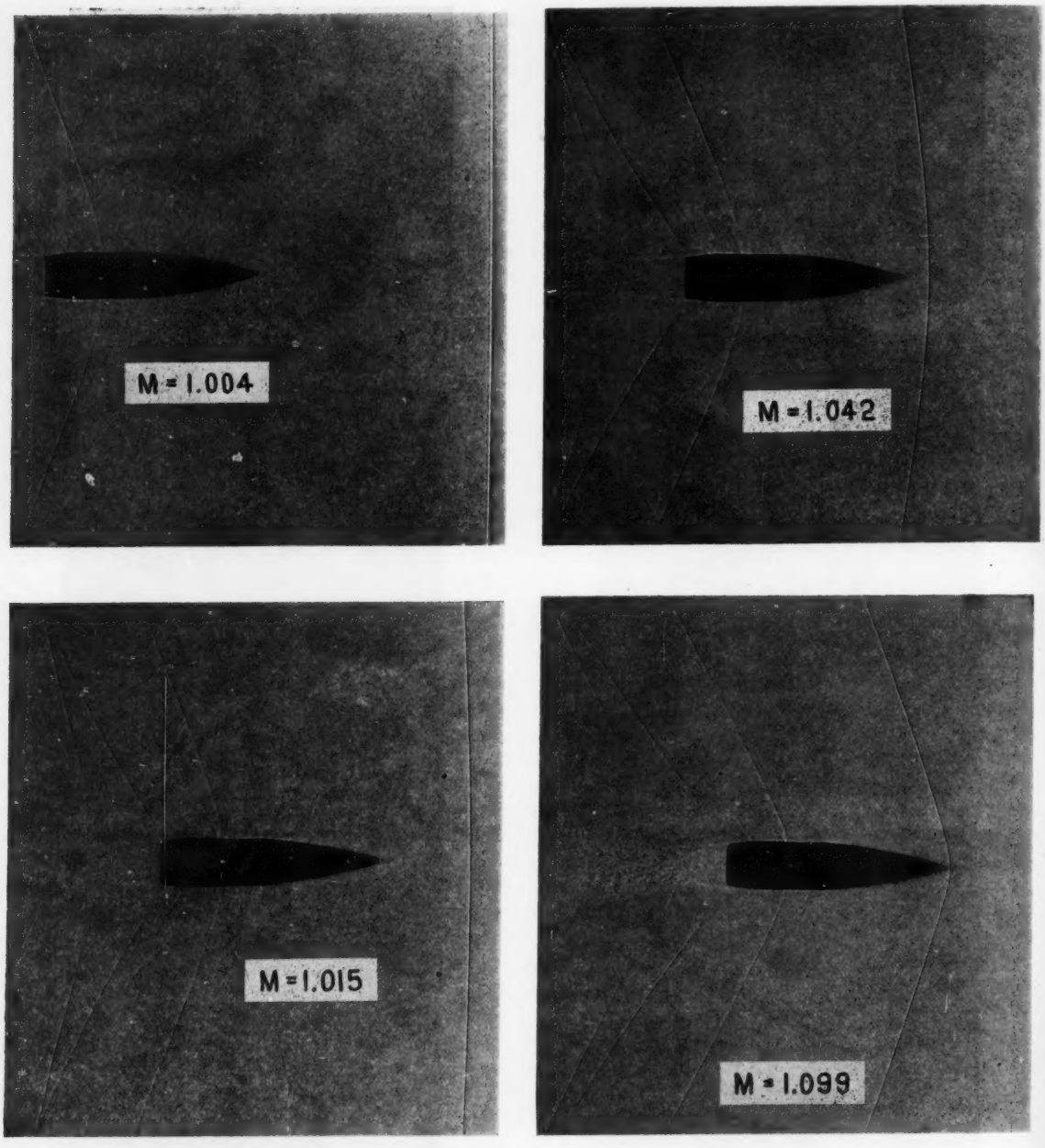
(c) The principal importance of the phenomenon of the shock wave, from the point of view of the practical aerodynamicist, is the drag caused by its presence, which was not revealed by the linear theory. I believe that the French mathematician, Hadamard, first showed that the flow of a gas passing through a curved discontinuity surface is not vortex-free, even in the case of uniform parallel flow ahead of the shock wave. Consequently, if a moving body produces a shock wave, the body is followed by a wake equivalent to a loss of momentum. This loss of momentum represents a certain amount of drag similar to the wake drag caused by flow separation.

One can look at the phenomenon also from the point of view of entropy. The shock wave involves an increase of entropy, which means that the kinetic energy that has been transformed into heat content cannot be retransformed entirely into kinetic energy. Consequently, the air at a large distance behind the moving body has a higher heat content than the air at a large distance in front of the body. To produce this heat, work has to be done—i.e., propulsive force is necessary to maintain the stationary motion of the body.

The drag corresponding to the wake of the shock wave and the wave drag cannot always be easily separated in the exact theory of the drag of bodies moving with supersonic speed. Consider, for example, a two-dimensional airfoil, and assume an attached shock wave at the sharp leading edge. If one draws the Mach lines starting from the surface of the airfoil, one notices that they intersect the shock wave. The Mach lines starting from the surface represent expansion waves of the type indicated previously in connection with the flow of a compressible fluid around a corner. They are sometimes called Prandtl-Meyer waves, since the mathematical solution of the expansion process was first given by these authors. As the expansion waves intersect the compression shock wave, they reduce its intensity and may also create infinitesimal compression waves reflected from the shock wave. By these processes the compression wave gradually loses in intensity, and at infinity a simple Mach line remains, analogous to the Mach lines encountered in the linear theory. However, one sees from this qualitative discussion of the exact process, that the expansion waves starting from



FIGS. 23. Shadow photographs (on this and following pages) of 155-mm. projectile in free flight through sonic velocity. (Released by courtesy of Ballistic Research Laboratory, Aberdeen, Md:)

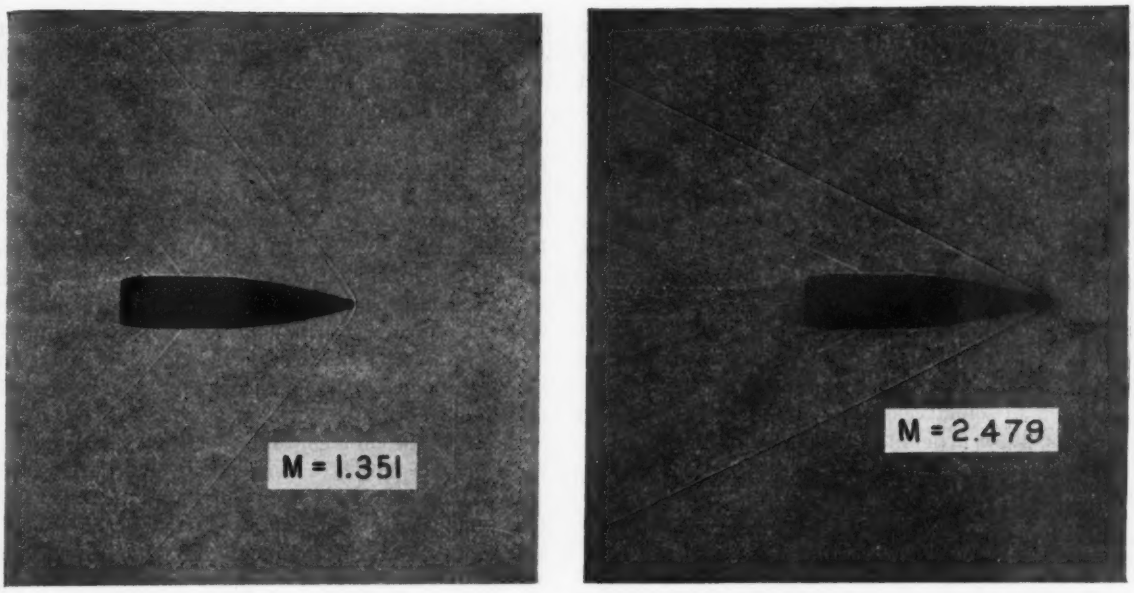


FIGS. 23. Continued.

the surface of the airfoil do not extend to infinity, and therefore they cannot contribute to the wave drag, which was defined as the equivalent of the momentum transferred to infinity. The wave drag is transformed into wake drag—i.e., into a drag due to the shadow of the shock wave.

However, the comparison of the calculated wave drag and the drag actually measured shows that, in spite of this difference in the physical nature of the drag mechanism, the concept of the wave drag leads to a fair approximation. The reason is that it represents correctly the conditions at a large but finite distance from the body. The case is somewhat analogous to that of the induced drag in the subsonic three-dimensional wing theory. The plane vortex sheet behind the wing assumed by the linearized subsonic theory cannot extend to infinity. Nevertheless, the computation of the induced drag based on this assumption gives a good approximation

A further approximation beyond the linearized theory can be obtained—at least for the case of two-dimensional or axially symmetric flow—by using the exact



FIGS. 23. Concluded.

flow equations instead of the linearized ones but neglecting the reflection of the wavelines at the shock wave. This method was applied by several authors to the calculation of pressure distributions on airfoils. One interesting result of this analysis—not revealed by the linear theory—is that the angle of deflection of the flow along an airfoil surface is limited. The fan-like expansion waves represented in Fig. 25 cannot be continued indefinitely, for at a certain angle of deflection zero pressure is reached. The same phenomenon occurs on the cambered surface of an airfoil. If the limit is reached, flow separation must occur. This phenomenon has important bearing on the maximum lift of wings and especially on the limitation of control surface effectiveness in supersonic flight.

In the case of a detached shock wave, a great portion of the flow in the neighborhood of the body may be subsonic. In this case the main part of the drag corresponds to the wake of the shock wave. The same is true in the case of transonic flow with shock wave of finite length. Suppose the velocity of the body is less than sound velocity but the relative flow around the body is partially supersonic. If a shock wave develops in such a case, it can only have a finite length, inasmuch as no shock wave can exist in the subsonic region. Since the flow is subsonic at a large distance from the body, certainly no wave drag can exist. The drag due to shock wave and separation, being in general much larger than the frictional drag, causes a considerable rise in the value of the coefficient of the total drag. The value of the Mach Number at which this rise of drag coefficient occurs is called the critical Mach Number. The transonic problem is discussed in some detail in Sections 11 and 12.

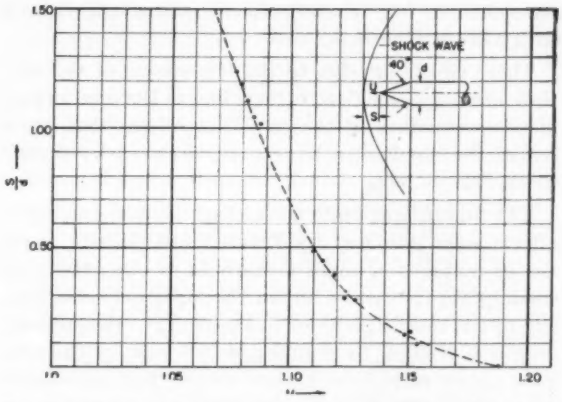


FIG. 24. The ratio between the distance of the detached shock wave and the projectile diameter as a function of Mach Number.

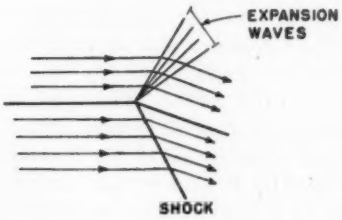


FIG. 25. Compression and expansion waves.

(d) The formation of shock waves is of fundamental importance in problems involving internal flows with supersonic velocity. The aeronautical engineer necessarily faces such problems in connection with the intake and ducting of the air required by propulsive devices.

One characteristic problem is that of the supersonic diffuser. The problem consists of decelerating air approaching a duct at supersonic speed to a speed that can conveniently be used in machines or combustion chambers. It appears that in this process shock waves cannot be completely avoided. The efficiency of the process can be essentially improved, however, by proper control of the location and magnitude of the shocks. The main design principle is avoiding normal shocks of great intensity. The aim of the designer is either to create a sequence of oblique shocks, such that the change of the normal velocity component is small at every step, or to produce weak normal shocks at sections where the velocity is only slightly greater than sound velocity. The scope of this paper does not allow a detailed discussion of this problem. Reference is made to work done by Oswatitsch, Crocco, and others.

(11) The Transonic Problem

The main effects of compressibility encountered in practical subsonic airplane design are: increase of drag; breakdown of lift; and, as a consequence of the second phenomenon, loss of control and maneuverability of wing-tail combinations.

These effects are due to the breakdown of the continuous vortex-free flow, rather than to the direct effect of compressibility of the air. The effects that occur before the breakdown have been analyzed by several methods.

The first approximate theories of the compressibility effects were proposed by Prandtl and Glauert, using the assumption of small perturbations—i.e., the same assumption that was made in the supersonic case leading to the results given in this paper. In the subsonic case it is possible to deduce a simple rule for the compressibility correction known as the Prandtl-Glauert correction. This amounts to the statement that, in the case of thin airfoil sections, the pressures acting on a surface element are to be multiplied by the factor $1/\sqrt{1 - M^2}$, where M is the Mach Number of flight. It soon appeared that this correction was not sufficient when the Mach Number was increased beyond 0.5 or 0.6. A more adequate correction was obtained by H. S. Tsien and the author. This correction does not use the assumption of small perturbations. It is based on a different linearization of the flow equations originally proposed by P. Molenbroek and A. Chaplygin and applied by several authors, especially to the problem of gas jets. By a modification of the original method of Chaplygin, H. S. Tsien and the author made the method applicable to problems of high-speed flow and especially useful for the calculation of forces acting on airfoils. The mathematical simplification is attained by replacing a portion of the adiabatic curve in the pressure vs. volume diagram of the gas by a straight line. The Kármán-Tsien correction gave satisfactory results in a speed range in which the Prandtl-Glauert correction did not seem satisfactory. However, neither of the two methods can give a correct answer if the local velocity at some point on the airfoil reaches sound velocity. For this case, the theory must be reconsidered. It is the belief of the author that it will be difficult to find a simple unified method that remains valid in the speed range in which the flow is partially supersonic. It may be possible to find useful approximations for the pressure distribution by combination of typical subsonic and supersonic flow patterns. Extrapolations without safe theoretical foundation, however, although in accord with the measurements in a few cases, are bound to be inadequate if the experimental material is further extended.

Consider the problem of a symmetric airfoil in uniform parallel flow of a nonviscous compressible fluid, and assume that the velocity of the undisturbed flow is gradually increased. Up to a certain Mach Number there will be only one unique solution of the flow equations, the velocity at every point being subsonic. At a certain Mach Number, sonic velocity is reached at some point, and it can be shown that this necessarily happens on the surface of the airfoil. Contrary to some previous opinions, it appears that the occurrence of sonic velocities does not necessarily mean that a continuous flow is not possible and that the airfoil has a resulting drag. The solution, in fact, can be composed of a subsonic and a supersonic domain without discontinuity. The supersonic domain is adjacent to the airfoil surface. If the Mach Number is further increased, at least at one point an infinite acceleration will occur, and beyond this limiting Mach Number no continuous solution is possible. Beyond this limit the airfoil must have a drag even in a nonviscous fluid. This drag is caused by the presence of shock waves, by separation, or by both. From the theoretical point of view, this limiting Mach Number should be considered as the critical Mach Number. There are several doubtful aspects of the question, however:

(1) It has not been shown that the continuous solution composed of a subsonic and supersonic domain is the unique solution.

(2) It has not been shown that a discontinuous solution—i.e., a flow with shock wave—cannot exist below the theoretical critical Mach Number. Certainly no such solution is possible before the velocity of sound is reached locally. However, the behavior of the fluid between these two limits has not yet been sufficiently explored theoretically, and the experimental evidence is also somewhat spotty. At the present state of knowledge it seems to be appropriate to call (a) the Mach Number at which the local velocity reaches sound velocity, the *lower critical Mach Number;* and (b) the Mach Number beyond which no continuous flow is possible, the *upper critical Mach Number*. The condition (a) gives a necessary, the condition (b) a sufficient, condition for the occurrence of a shock wave.

(3) It can certainly be stated that the actual breakdown causing a rapid rise in drag and a drop in lift must occur between the lower and upper critical limits. In no case has it been shown that the upper limit was attained experimentally. A major handicap in such investigations is the difficulty of finding the upper limit theoretically. Moreover, according to a classification by H. S. Tsien, the following factors may, in the case of such large accelerations as necessarily occur near the upper critical limit, give appreciable alterations in the play of the dynamic forces: (a) viscous stresses due to ordinary internal fluid friction; (b) viscous stresses due to rapid compression and expansion; (c) relaxation lag in internal molecular vibrations; and (d) heat conduction.

The author believes that especially instability of the continuous flow can lead to a premature appearance of shock waves.

The experimental evidence also shows that the boundary layer has considerable influence on the formation of shock waves. This problem is known as that of the interaction between boundary layer and shock waves. It appears necessary to devote at least a short discussion to this question.

(12) Interaction Between Boundary Layer and Shock Wave

The concept of the boundary layer is based on the assumption that the flow conditions in the boundary layer, its growth or decrease, are fully determined by the velocity and pressure distribution of the nonviscous outside flow. In other words, one assumes that the flow outside of the boundary layer influences the development of the boundary layer but that there is no influence of the boundary layer on the main flow. In the case of an incompressible fluid or a compressible fluid moving at low velocity, it can be shown that this statement is really true unless the flow separates. The separation can be caused by a sharp corner or by a large magnitude of the so-called adverse pressure gradient. Unless there is separation, the pressure across the boundary layer is sensibly constant and changes in thickness of the boundary layer do not influence the main flow appreciably.

The fundamental assumption of the boundary layer theory cannot be applied to a flow in the transonic speed range for two reasons:

(a) In the neighborhood of sonic velocity a slight cross-section change causes large changes in pressure and velocity. It is known that, in a flow through a supersonic nozzle, sonic velocity occurs at the minimum cross section, so that both slightly smaller or slightly larger velocities require a larger cross section. This fact indicates that a flow near sonic velocity must be extremely sensitive to changes in cross section. Therefore, if one considers the streamline tubes passing just outside of the boundary layer, one must conclude that increase or decrease of the boundary-layer thickness must have an essential influence on the flow in the neighborhood by narrowing or widening the cross sections of the neighboring streamline tubes. Consequently, near the velocity of sound there is interaction between the main flow and the boundary layer in both directions.

(b) Since the flow in the boundary layer, at least near the wall, is certainly subsonic, no shock wave can extend through the whole boundary layer and end at the wall. Now if the shock wave ends at some point inside of the boundary layer, the conditions that prevail must contradict the assumptions made in the boundary-layer theory. The pressure discontinuity at the shock wave must create a steep pressure rise in the subsonic part of the boundary layer. The pressure across the boundary layer will no longer be constant, and therefore the ordinary boundary-layer theory can hardly be applied. The large pressure rise may cause separation, and the separation will, in general, react on the direction and magnitude of the shock wave.

The importance of the interaction between the boundary layer and transonic flow, especially the formation of shock waves, was recognized at about the same time by scientific workers at the N.A.C.A., at the California Institute of Technology, and by Professor Ackeret in Zurich. The research on this subject is still far from being finished. Several important findings can, however, be mentioned. It has been recognized that the formation of shock waves on a cambered airfoil surface depends, to a large extent, on the type of flow in the boundary layer—i.e., on whether it is laminar or turbulent. In the case of laminar boundary layer, a wave combination is observed which, by its configuration, resembles the Greek letter λ. It appears that the increase in thickness of the laminar boundary layer produces in the main flow a system of oblique waves or weak shocks, apparently followed by one strong shock. The photographic evidence and the evidence obtained by measurements of the wall pressure distribution seemed contradictory at first. In spite of the large intensity of the shock wave apparent from the photographs, almost no pressure difference was measured at the wall. This discrepancy was cleared up by improvement of the technique and a more exact analysis of the photographs. H. Liepmann, of the Guggenheim Laboratory in Pasadena, found that the shock is followed by a fan-like system of expansion waves. As the shock is inclined in the upstream direction and the expansion waves are inclined in the downstream direction, this phenomenon can be considered as reflection of a compression wave at a free boundary, which in this case is the subsonic part of the laminar boundary layer. It is known that a compression wave is reflected at a fixed wall as compression and at a free boundary as expansion. As was mentioned before, no expansion shock can exist, and therefore the reflected wave appears as a fan-like system of wave fronts. Photographs il-

lustrating these phenomena can be found in the papers of Liepmann and Ackeret.

If the boundary layer is or has artificially been made turbulent before the maximum velocity is reached, a strong compression shock is shown in the photographs. The shock is approximately normal to the wall and the wall pressure measurements show a rapid pressure rise caused by the presence of the shock wave. Sometimes the shock wave has a slight inclination to the flow direction, probably because the main flow is deflected by separation or by rapid increase of the thickness of the boundary layer. In such cases some reflected expansion waves are also visible, as in the laminar case, although the reflection is not sufficient to eliminate the pressure rise at the wall.

It has been known for a long time that in subsonic flow critical changes in the drag and stalling characteristics of airfoils depend on the relative location of the transition point between laminar and turbulent flow and the separation point. It is interesting to note that the same interplay between transition and separation also determines the formation of shock waves and therefore the drag and lift characteristics of airfoil sections and slender bodies at transonic speed. However, we are only at the beginning of an understanding of the phenomenon and have not yet reached the stage of practical conclusions.

(13) The Theory of Transonic Flow

In recent years great efforts have been made to compute the flow around bodies of various shapes in the transonic speed range, especially to construct mixed solutions of the flow equations consisting of supersonic and subsonic domains. One finds experimentally that the supersonic region expands rapidly when the Mach Number approaches unity. It was found experimentally, in the case of an airfoil with 12 per cent thickness ratio, that the size of the supersonic region normal to the chord increased from zero to 28 per cent of the chord length when the Mach Number of the flow increased from 0.795 to 0.844, and reached 46 per cent of the chord length at $M = 0.875$. The corresponding values of the maximum local Mach Number on the airfoil surface were 1, 1.147, and 1.189, respectively. Theoretical calculations have been successful in some simple cases. They require, however, a large amount of analytical and numerical work.

The linear perturbation theory is not valid for the transonic range. In order to obtain a simplification of the flow equations they must be considered from a new point of view. It has been said that the linearized theory is based on the assumption that all velocities produced by the presence of a moving body are small in comparison to both the velocity of flight and the velocity of sound. However, if one follows exactly the process of deduction of the linearized equation, one notices that the assumption was made that the velocity

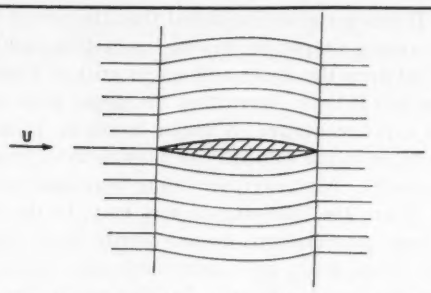


Fig. 26. Flow at sonic velocity according to the linearized theory.

perturbations must also be small in comparison with the difference between flight and sound velocities. In the transonic case, this assumption does not apply and must be replaced by the assumption that all velocities, including the velocity of the main flow, are only slightly different from the velocity of sound. The breakdown of the linearized theory is illustrated by the obviously incorrect result that it yields for $M = 1$, shown in Fig. 26. It is seen that all streamlines have identical curvatures. According to elementary laws of dynamics, an infinite pressure difference should therefore exist between the surface of the airfoil and infinity. The new assumption leads to a simplification of the flow equations and promises to lead to useful results with a moderate amount of mathematical work. Besides that, it gives a simple rule of similarity for transonic flow around bodies and wings with similar thickness, camber, or angle of attack distribution. For example, in the case of two-dimensional flow around a symmetric airfoil, one obtains the rule that, in order to have a similar flow condition around airfoils with similar thickness distribution, one must change the quantity $1 - M$—i.e., the difference between unity and the actual flight Mach Number, in proportion to the two-thirds power of the thickness/chord ratio.

According to this rule, the experimental drag curves for airfoil sections with different thickness ratio are reduced to one single curve according to the equation

$$C_D = \frac{1}{M^2}\left(\frac{t}{c}\right)^{5/3} D\left[\frac{(t/c)^{1/3}}{\sqrt{1 - M^2}}\right]$$

It seems that this rule is confirmed by the few experiments that are available in the transonic range. The lift coefficient of an extremely thin (flat plate) airfoil should follow the law expressed by the equation

$$C_L = (1/M^2)\, i^{3/3}\, L(i^{1/3}/\sqrt{1 - M^2})$$

where i denotes the angle of attack. If the similarity rule applies at sound velocity, the drag of a symmetric two-dimensional airfoil at sound velocity is proportional to the five-thirds power and the lift coefficient of a thin airfoil to the two-thirds power of the thickness ratio and the angle of attack, respectively.

The problem of transonic flow raises many interesting questions that have not yet been decided either experimentally or theoretically. For example, it is questionable whether a stationary solution for the flow past a body is possible when the free-stream velocity is assumed to be exactly equal to sound velocity. It is evident that no such solution can exist if the stream is limited between parallel surfaces. Another problem, perhaps more important practically, is that of the accelerated or decelerated motion in the transonic range and at the velocity of sound. In the theory of incompressible fluids, the apparent mass of a body gives a measure of the air mass involved in the acceleration caused by the motion of the body. Now it is reasonable to assume that the corresponding effect increases rapidly when the velocity approaches the velocity of sound. The problem of accelerated motion has several practical applications. It enters, for example, into the theory of wing oscillations and flutter. It also enters into the evaluation of dropping experiments, undertaken for the determination of aerodynamic characteristics of bodies in the transonic range, to replace wind-tunnel measurements, which become questionable in the transonic range.

However, from a practical point of view the most important questions remain: the determination of wing shapes, plan forms, and sections which cause a postponement of the critical phenomena—namely, the rise of drag and breakdown of lift at Mach Numbers approaching unity. It is known that one important method is the use of large sweepback. The fundamental idea of this method is the decrease of the effective Mach Number, which is assumed to be equal to the Mach Number corresponding to the component of the flight velocity normal to the leading edge. The detailed exploration of transonic phenomena on wings with sweepback is also important for supersonic flight, since the sweepback produces transonic conditions over certain parts of the wing even at Mach Numbers much larger than unity.

(14) A Simple Method of Range Prediction for Supersonic Airplanes

According to Breguet's formula, the range of an airplane depends on three quantities: (a) the consumption of propellants carried in the airplane and necessary to produce a unit amount of useful work, including thermodynamic, mechanical, and propulsion efficiency; (b) the lift/drag ratio of the airplane; and (c) the ratio of the initial flight weight of the airplane to the weight that remains after all the propellant is gone.

The first quantity depends on the future developments of propulsive units and will not be discussed in this paper. The third one is essentially a question of structural design and is also outside of our considerations. The second parameter—namely, the lift/drag ratio of the entire airplane—is primarily a problem of aerodynamics. In the case of supersonic airplanes, the aerodynamic efficiency depends to a great extent on the solution of the problem of volume—i.e., on the ability of the airplane designer to provide volume for fuel or propellants. Of course, the same point of view occurs also in the design of long-range transportation airplanes in the subsonic range. In supersonic flight the influence of the volume requirement on the aerodynamic efficiency of the plane is greatly exaggerated by the high wing loading. In other words, the supersonic airplane easily becomes a large body with small winglets, and, correspondingly, the lift/drag ratio becomes rather unfavorable.

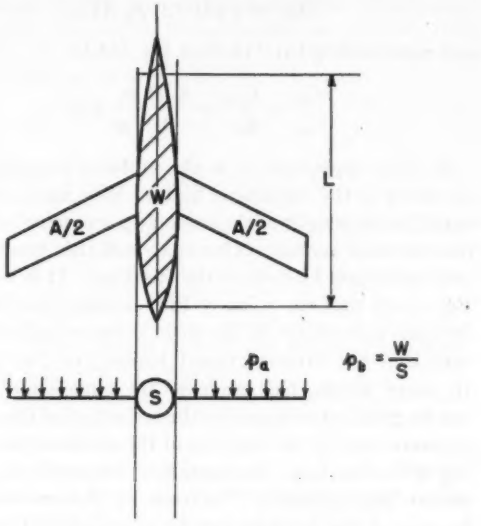


FIG. 27. Wing-body combination.

The following simple calculation is crude but may illustrate the point (see Fig. 27). Denote the wing area by A; the master cross section of the fuselage by S; the drag and lift coefficients of the wing referred to the wing area by C_{Dw} and C_L, respectively; the drag coefficient of the fuselage referred to the master cross section by C_{Df}. Then the resultant drag/lift ratio of the airplane is equal to

$$C_D/C_L = (C_{Dw}/C_L) + (C_{Df}S/C_LA) \qquad (14.1)$$

Now one can express the condition that the lifting force be able to carry the weight of the body (the weight of the wing is neglected). Assume that the weight of the body is equal to the product SLw, where L is a kind of reduced length of the fuselage and w is the average specific weight of the fuselage full of pay load and fuel. Thus we have the equation

$$SLw = \rho(U^2/2)AC_L \qquad (14.2)$$

Taking into account the definition of the Mach Number and the formula for the velocity of sound, one obtains

$$SLw = (\gamma M^2/2)\, p_a\, A C_L \tag{14.3}$$

and substituting Eq. (14.3) in Eq. (14.1)

$$\frac{C_D}{C_L} = \frac{C_{Dw}}{C_L} + \frac{\gamma M^2}{2}\frac{p_a}{Lw} C_{Df} \tag{14.4}$$

In these equations p_a is the ambient pressure corresponding to the altitude of flight. Now Lw is obviously equal to the weight of the fuselage per square foot of the master cross section. One may call this quantity the cross-sectional loading of the fuselage. It is seen from Eq. (14.4) that the value of the resulting drag/lift ratio depends essentially on the ratio between ambient pressure and the cross-sectional loading of the fuselage. In other words, the range of the supersonic airplane can be greatly increased by the reduction of the ambient pressure and by the increase of the cross-sectional loading of the fuselage. Reduction of the ambient pressure means high altitude. Increase of the cross-sectional loading of the fuselage can be accomplished by denser loading, by use of high specific density fuels, and, finally, by the increase of the overall dimensions of the airplane. Obviously, the weight increases with the third power and the master cross section with the second power of the linear size. Consequently, the cross-sectional loading of the fuselage, everything else remaining similar, will increase in proportion to the linear dimensions of the airplane.

The results expressed by Eq. (14.4) are remarkable in that, for the conventional subsonic speed range, flight altitude has no significant effect on the range of airplanes. Velocity and size also have only secondary influence. The effect of flight velocity on range in the supersonic case is expressed in Eq. (14.4) by the quantity $(\gamma/2)M^2C_{Df}$; this product has a flat minimum between $M = 1.5$ and $M = 2$.

If one proceeds to extremely high altitude, the wing loading decreases essentially and the proportion between wings and fuselage becomes more conventional. Therefore, for large dimensions and extremely high altitudes, the flying wing may again have fair possibilities, and the analysis must include the volume available in the wing, which was neglected in the above considerations.

The application of flying wings of usual design in supersonic flight may be restricted by the necessity of employing thin sections at supersonic speed. On the other hand, when large sweepback is used, this necessity is no longer imperative and allows more freedom in the choice of sections. The triangular wing also appears suitable for flying wing design and may combine small thickness/chord ratio with relatively large included volume.

The author does not want to commit himself to numerical range predictions on the basis of the approximate method presented in this section. In a real performance calculation, the period of climb and acceleration through the transonic speed range must be included.

The reader himself may substitute numerical values that he believes are realistic enough to give answers that are more than guesses. The author will be satisfied if his general considerations have given some inspiration to further developments in supersonic aerodynamics.

The author is glad to acknowledge the assistance and aid he obtained from a number of friends and younger scholars with whom he discussed questions pertinent to this paper. They were extremely helpful not only in preparation of diagrams and calculations but also in clearing up several fundamental aspects of the problem discussed. He wants to mention especially H. S. Tsien, W. Hayes, C. C. Chang, and W. Perl. A. C. Charters and H. W. Liepmann kindly contributed some results of their research to Sections 10 and 12. The considerations given in the last section resulted from the author's discussions with H. L. Dryden and G. B. Schubauer. For some of the bibliography I am indebted to R. C. Roberts.

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Discussion of the Lecture

Dr. Francis H. Clauser, Chairman, Department of Aeronautics, Johns Hopkins University: "For the Tenth Wright Brothers Lecture we have had the pleasure of having one of the aeronautical world's most distinguished scientists and engineers present one of his justly famous reviews of a field of scientific endeavor. For his lecture he has chosen to discuss the field of supersonic aerodynamics, and the size and eminence of the audience attest the great interest now being shown in this field. I consider myself fortunate to have had the opportunity of working with Dr. von Kármán more than 10 years ago on problems in compressible flows. During the intervening years it has been interesting to watch the remarkable growth in importance of supersonic aerodynamics and the equally impressive growth of esteem and respect for Dr. von Kármán.

"In the thirties Dr. von Kármán had securely established a world-wide reputation as an aeronautical scientist. However, few in American industry were ready to accept him as a prophet of scientific engineering whose creed was progress through physical understanding, backed by the potent tools of mathematics. During this same period there were only a few evidences of interest in the effects of compressibility.

"As we entered the war era, speeds mounted rapidly, and we soon found ourselves struggling to obtain an understanding of the various phenomena associated with the compressibility of air. With the advent of jet and rocket power plants, our thoughts turned to the possibilities of transonic and supersonic flight. Speeds such as these brought with them problems that could not be solved by the intuitions that had served engineers so faithfully. Instead it became necessary to follow what today might be called the 'Kármán Doctrine.' The fact that Dr. von Kármán was asked to deliver the Tenth Wright Brothers Lecture, the fact that he has chosen to speak on supersonic aerodynamics, and the fact that his lecture has had such an outstanding attendance, all attest to the success of this doctrine and its prophet.

"In the past, Dr. von Kármán has established a pattern in his principal lectures of making at least one outstanding contribution to the field under discussion, in addition to his now well-known résumés in which he gives a picture of the broad physical concepts that serve as stepping stones to further progress in the field. I believe there is little doubt that the outstanding contribution of this lecture is his method of Fourier integrals. Already Dr. von Kármán and his coworkers have solved an impressive number of important cases by this method. As one considers the method carefully, it becomes clear that it is a powerful tool and undoubtedly will be extended to solve such problems as the interaction of wings, tails, and fuselages, as well as other complex problems.

"I cannot help expressing a feeling of regret that in this lecture Dr. von Kármán has dealt so little with the nonlinear problems of compressible flows. There is no reasonable doubt that the linearized solutions have provided both physical understanding and engineering data at a time when both are sorely needed. However, a growing number of people (including Dr. von Kármán himself) believe that many of the problems urgently in need of solution are inherently nonlinear in character. In these problems, aeronautics will be making independent contributions in the forefront of the other sciences. Along with others, I express the hope that Dr. von Kármán may soon again grapple with some of the nonlinear problems.

"At this point, I should like to put forth a question about interpretation. In section 11, on the transonic problem, the lecturer, following Tsien, defines an upper and a lower critical Mach Number. The lower value corresponds to the first appearance of local velocities equal to the velocity of sound. The upper value corresponds to the limit beyond which no continuous flow is possible. Undoubtedly such an upper limit exists, particularly for real fluids. There is little reason to doubt its existence even with nonviscous fluids, although the existence of the upper limit has never been established rigorously for nonviscous fluids to the best of my knowledge. To show the likelihood of the existence of such a limit, the lecturer states that 'if the Mach Number is further increased (beyond the first critical Mach Number), at least at one point of the surface an infinite acceleration will occur, and beyond this limiting Mach Number no continuous solution is possible.' Actually, all solutions obtained so far which have demonstrated this tendency have been solutions in which the geometrical boundaries distort with Mach Number in such a way that at the limit mentioned above an infinite curvature of the airfoil surface appeared at the point of infinite acceleration. This is clearly not an answer to the problem of the upper limit for an airfoil of fixed contour whose curvature is finite. Physically, it is conceivable that a certain class of airfoils may have an upper limit of this type in which incoming compression waves coalesce at the surface (and are reflected in order that the curvature remain finite). However, airfoils with fore and aft symmetry would appear not to be members of this class. All such nonmembers, if they possess upper limits, will have such limits caused by something other than the appearance of points of infinite acceleration at the surface. All of the above remarks apply to nonviscous fluids for which we must obtain answers as to the existence and

uniqueness of flow patterns before we can hope to understand the problems of real fluids.

"In closing, I should like to extend my warmest congratulations to Dr. von Kármán for having presented the wealth of material that is contained in his Tenth Wright Brothers Lecture."

Capt. Walter S. Diehl, Engineering Division, Bureau of Aeronautics, Navy Department: "Dr. von Kármán is a great scientist and educator, in addition to being one of the world's foremost aeronautical engineers. He was a real pioneer in the study of compressibility and supersonic flows. Under these circumstances anything that he may have to say is certain to be significant and worthy of most careful attention.

"This lecture is particularly significant in many respects. It is a most timely one. It may be described as a major effort to collect all of the existing knowledge on supersonic flows into a systematic pattern fully coordinated, compared, and contrasted with subsonic flows. This has been done with a degree of clarity and conciseness that borders on being deceptive. Many of the items are far from being as simple as Dr. von Kármán now makes them appear. Those who follow the theoretical progress in supersonic aerodynamics are fully aware of the tremendous background of work involved in this summary.

"It may be noted that the qualities of clarity and conciseness seem to characterize all of Dr. von Kármán's papers. He has a most unusual ability to present his essential, completed equations in forms that always look familiar, even when we know that something entirely new and a major contribution to aeronautical sciences is involved.

"No one is likely to criticize the paper on the ground that it makes the subject appear too simple. This would, of course, be contrary to the popular ideas regarding the difficulties encountered in the mathematical analyses of supersonic flows. Nor is there likely to be any criticism on the ground that Dr. von Kármán has ignored these difficulties, since he points out clearly where they remain. There are still enough of them to keep the theoretical men at work for many years to come.

"The lecture contains few statements to which an engineer can object. On the other hand, the mathematician may find several of the assumptions lacking somewhat in the rigor that is necessary to satisfy all conditions. The mathematician may also protest mildly over certain statements in the lecture, such as the statement that the center of the pressure area created on the ground by an airplane flying at low subsonic speeds lies vertically under the center of gravity of the airplane. He will point out that the distortion of the ground pressure pattern follows a definite law applicable at all speeds so that the center of the pressure area never lies vertically under the center of gravity of the flying airplane. It could be claimed that neglect of this fundamental distortion of pressure patterns may prevent the full understanding of the significance of associated phenomena and lead to overemphasis of certain features of supersonic flows. Such criticisms, though valid, are not really important. The important things here are the principles and applications of supersonic aerodynamics so clearly set forth by Dr. von Kármán. The fact that these principles and applications are presented without frills will be most welcome to all aeronautical engineers. It will come to many as a pleasant surprise to find that so much is already known regarding flow conditions at supersonic speeds. All of those who work with supersonic aerodynamics owe a debt of gratitude to Dr. von Kármán for his lecture."

F. W. Geiger, Head, Theoretical Aerodynamics Branch, Cornell Aeronautical Laboratory: "Dr. von Kármán has long been noted not only for his very considerable contributions to the Aeronautical Sciences but also for the wonderful way in which his papers are written and presented. Certainly his reputation for such things has in no way suffered today. In fact, in my estimation the paper presented today is such a masterpiece of technical writing that I wish to confine my remarks to this subject.

"All too many of us feel that once an idea has been developed into what we consider to be a consistent theory the work has been done. We tend to forget that the method of presentation of the information is just as important as the development itself. We spend as little time organizing the paper as possible so as to get the job done in a hurry. That we do not proceed logically from step to step, from part to part, does not seem to matter too much. The listener will get the idea. It does not matter that we do not state what we are doing or where we are going. The listener will find out when we get there. We concoct new words or give old words new definitions, too frequently only giving the impression that we do not know what we are talking about. And we tend to use big words. Somehow these sound 'learned.'

"So it is refreshing—to say the least—to listen to a lecture such as this one. The lecture has been organized so effectively that the details of that organization are self-evident. The coherence between the parts of the paper leaves nothing to be desired. As each new subject is discussed, it is completely obvious what is being attacked and by what means it is being attacked. And the wording is apparently the simplest consistent with accuracy, in Dr. von Kármán's vocabulary. He must have spent hours breaking down paragraphs, sentences, and phrases to as simple and as easily understandable terms as possible.

"Today we have heard and seen the results of inspiration and perspiration—in such proportions as to define genius. In my opinion, most of us could well afford to lose more perspiration. For it is probably true that perspiration is responsible for most inspiration."

Col. Paul H. Kemmer, Chief, Engineering Plans, Engineering Division, Wright Field: "In the first part of this excellent lecture, Dr. von Kármán described how Sir Isaac Newton has succumbed to the use of an alibi to explain some variance between his theory and experiment. Where unknowns are involved, it would seem proper to assume that an alibi offered by a discusser should be accepted as graciously as one presented by a genius. Although it was necessary to prepare this discussion before the preprint of the complete paper was available for study, I hope the brief comments that follow will be appropriate.

"Dr. von Kármán's paper on supersonic aerodynamics is most timely. High-speed performance continues to be a major factor in flight requirements. Much of the technical effort in the institutions of higher education, in the aeronautical and parallel industries, and in the pertinent governmental agencies is devoted to problems involving supersonic aerodynamics. The requirement for a good knowledge of supersonic flow has existed for some years. However, the trend of aeronautical development is, and will be, such that an ever-increasing proportion of the personnel engaged will be required to have a clear picture of the principles of supersonic aerodynamics. Only a few years ago, the problems in supersonic aerodynamics were limited, with a few exceptions, to those attending local flow disturbances in subsonic flight. Today, however, a large cross section of the aeronautical world is working diligently on the problems involved with transonic and supersonic flight. Our progress in this rapidly expanding field of endeavor depends upon the accurate conceptions and clear thinking on the part of the technical people engaged. This important contribution by Dr. von Kármán should prove most helpful to this progress and to those striving for a clear picture of the principles of supersonic aerodynamics. The physical concept and the similarity and the momentum concepts, as well as the treatment of the relationship between the shock wave and drag, are important contributions at this time.

"The pronounced shift of effort to transonic and supersonic flight problems started comparatively recently and after nearly one-half century was spent on subsonic problems. During this period the importance of aircraft in war and peace was established. This shift of aeronautical effort has been accelerated greatly by the development of jet propulsion systems that afford the essential increase in thrust. Atomic energy and new fuels suggest greater specific thrusts for propulsion systems. Thus, the aeronautical progress so far appears to have developed merely the threshold of a new era, with unlimited possibilities and with supersonic craft predominating. Considering our progress in that light then, it would seem reasonable to assume that our best picture of the principles of supersonic aerodynamics is far from complete. Possibly, some important factors and relationships required for that complete picture are unknown today. These unknowns must be discovered through fundamental research and diligent development. The importance and tempo of aeronautical progress demands the discovery of the unknowns as rapidly as possible. The proper accomplishment of this work will require that the agencies primarily responsible for research devote continuously their entire efforts and facilities to fundamental research. Real progress in research requires concentration on that work. Diversionary work, such as development or evaluation testing of specific equipment, is not conducive to the attainment of the necessary results to be expected from fundamental aeronautical research. The airfoils now being suggested for supersonic flight must be considered stopgaps in the development of transonic craft. Research should point the way to the profitable use of shock waves in propulsion systems, as well as in aircraft and missiles. In proceeding from this threshold, the responsibilities of the research agencies will not be fulfilled unless those agencies devote their efforts entirely to basic research. The application of the fundamentals of research, through applied research and development, is as important as research itself. This statement is based upon the many experiences where important research was suggested only through the applications or developments. There is a mistaken conception by some people that the research and the facilities therefor should have priority over development and evaluation. As in the past, it is still logical, and it will be so in the future, that research without application is useless, and development and evaluation testing, in many cases, are stimulant and catalyst for research that would not be started without them.

"Dr. von Kármán has been working closely with personnel of the Army Air Forces since 1931. During this period, he has made many important contributions that are reflected in the excellence of the aeronautical matériel. His advice has served well to attain the desired results with reduced experimentation where time was costly. His guidance in research and development has been invaluable. Dr. von Kármán has hosts of friends in the Army Air Forces who hold him in highest esteem, and this opportunity to express their gratitude to him is greatly appreciated."

John Stack, Chief of Supersonic Research, Langley Memorial Aeronautical Laboratory, N.A.C.A.: "The brilliant exposition of the theoretical basis of supersonic aerodynamics given by Dr. von Kármán gives the aeronautical engineer an unmistakable path. The lecture indeed parallels a paper of twenty-five years ago by Prandtl, 'Application of Modern Hydrodynamics to Aeronautics,' *N.A.C.A. Report No. 116.* Just as Prandtl's famous paper outlined the direction for the engineer in the development of subsonic aircraft, Dr. von Kármán's lecture outlines the direction for the engineer in the development of supersonic aircraft. Though Prandtl's paper appeared in 1921, several years elapsed before the design of aircraft gave evidence that the direction given had been heeded. We must not permit history to repeat.

"The extensive theoretical background of supersonic flows presented by Dr. von Kármán should not lead one to overlook the fact that major problems remain. Just as Prandtl's famous paper gave inspiration to, and was followed by, the theoretical work of Munk, Glauert, and many others, so also should we expect a similar development of the supersonic theory to follow the outline given by von Kármán.

"The lecturer noted that the aeronautical engineer 'should start to get the same feeling for the facts of supersonic flight as he acquired in the domain of subsonic velocities.' There can, of course, be no question with this view. That 'feeling,' however, may be long coming because the supersonic flow 'feels' different from the subsonic flow. For example, the boundary-layer thickening over the after portion of a two-dimensional supersonic wing reduces the wave drag and may, in some instances, result in the total of the actual wave plus skin-friction drag being less than the theoretical drag, a quite opposite 'feel' to that of the subsonic domain. The whole interference problem may likewise have a different 'feel' as a consequence of the opposite sign of the stream tube-area-velocity relation for sub- and supersonic velocities. Hence, for the immediate future at least, the aeronautical engineer may well discard his acquired subsonic 'feel' and be guided by a sound theoretical background to shapes and configurations that cause only the 'small perturbations' of the theory. In this respect it is well to note that our best subsonic aircraft are those having the small or negligible induced velocities for which the subsonic theory is applicable. It may very well be that the engineer will have but small use for the 'exact' theory that admits strong disturbances if the aircraft are to be efficient.

"Of the many unsolved problems the turbulent boundary layer appears to be one of great difficulty. Dr. von Kármán has pointed out that the problems of the theory of fully developed turbulent boundary layers and turbulent separation have not yet been solved for incompressible flows. As the shock is, in the first approximation, a severe adverse pressure gradient (theoretically infinite), it could well be that the next great contribution to the solution of trans- and supersonic flow problems may come from the boundary-layer research now under way at subsonic speeds. In view of the complexity of the turbulent boundary-layer problem, it may even be desirable to avoid the complication of the presence of shock in much of the fundamental research on the problem in the interests of obtaining a speedier solution.

"The transonic problem is, of course, a combination of the unsolved shock and boundary-layer problems. From the engineer's point of view, swept or delta wings appear to be the immediate answer. The effect of sweep really acts to raise the critical Mach Number, and with sufficiently great sweep and small wing-thickness ratio the critical Mach Number may be raised to supersonic values where again flow disturbances may occur. With the swept wing, however, the intense normal shock and associated boundary-layer disturbances common with straight wings in the transonic range appear to be replaced by very much reduced disturbances. An important consideration from the engineer's point of view is that, with highly swept wings, the critical Mach Number range will generally lie outside the 'choked' range of wind tunnels, so that laboratory methods are available and the more costly and time-consuming flight method may be required only as check means. Significant also in relation to the wind-tunnel method has been the development of new techniques that reduce limitations associated with the choking phenomena.

"The problem of the interaction of shocks and boundary layers, which has been extensively studied particularly over the last few years, as Dr. von Kármán notes, is still far from solution. It has been shown that the Reynolds Number has a great influence on the type of shock and the associated boundary layer disturbances. The Reynolds Number does not, however, appear to be the sole criterion. Lindsey has shown λ-type shocks and normal shocks at the same Reynolds Number dependent upon surface curvature. This problem is more important. Of great importance in relation to the research on this problem is the development of a theory of mixed flows even without the viscosity consideration. Such a theoretical development would permit a more definitive investigation of the boundary-layer effects. The assumption introduced by Dr. von Kármán and the similarity rule are an important contribution in this direction.

"The need for information on control-surface characteristics has been emphasized by the lecturer. In the absence of extensive design information on supersonic control surfaces for airplanes we must resort to the theory that shows us that, at certain deflections dependent upon the Mach Number, critical phenomena may occur as a consequence of reduction of Mach Number on the pressure side of the surface to

the value 1.0. In any experiments particular attention should be given to phenomena in this range.

"The Institute of the Aeronautical Sciences and Dr. von Kármán have done the advancement of aeronautical science a particularly great service in this brilliant and timely lecture. It is particularly appropriate as commemoration of the accomplishment of flight by the Wright Brothers."

W. Bollay, Director, Aerophysics Laboratory, North American Aviation, Inc.: "Supersonic aerodynamics represents, at present, one of the most fertile fields of investigation for the theoretical and experimental aerodynamicist. In this lecture, Dr. von Kármán has outlined to us the most interesting new developments in this field to which he has contributed so many of its basic ideas. He has provided all of us who are engaged in this field of research with stimulating new thoughts and a new perspective.

"Dr. von Kármán's periodic reviews of aerodynamics are classics that have served as the source of inspiration for much of the theoretical work that has been carried out. His last comprehensive review of supersonic aerodynamics at the Volta Congress in 1935 has served directly as the source of numerous important theoretical investigations. He suggested, for example, in 1935, the method of calculating the lift of a body of revolution which was later carried out by Tsien about 1938. At the same time, he suggested the method of calculating the wave drag of finite wings by sources and sinks, a method that has only been applied during the past year by Puckett. I feel sure that the suggestions made in his lecture today are going to be equally fruitful. I believe that particularly the method of Fourier integrals may prove to be a powerful tool in tackling the more difficult problems that cannot be handled by the method of conical flows or the method of singularities (using sources, doublets, and horseshoe vortices).

"I was again impressed by the common sense of many of Dr. von Kármán's practical recommendations: (a) The aeronautical engineer should start to get the same feeling for the facts of supersonic flight which he has acquired in the domain of subsonic flights. (b) The aerodynamic design of control surfaces constitutes a branch of supersonic aerodynamics which is very much in need of fundamental investigation. (c) The suggestion to apply boundary-layer control as a direct method of airplane control deserves careful study.

"In conclusion, I should like to say that I agree fully with one of my associates, who after reading Dr. von Kármán's lecture, stated: 'It reads exactly as if it had been written by Lord Rayleigh.' It shows the touches of the old master in every detail."

J. Kampé de Feriet, Professor, University of Lille; Honorary Director of the Fluid Mechanics Institute of Lille: "It is difficult to add anything to such a brilliant lecture; I confess that I do not know which I admire more, the science of Dr. von Kármán or his humor in presenting such important and difficult results.

"Having just arrived today from France by plane, I am not well prepared to discuss a paper that is not closely connected with my own field of research, but I think that you will be glad to hear some news from Paris. One item, which is perhaps not already known in this country, should be especially interesting for the Institute of the Aeronautical Sciences: the Académie des Sciences de l'Institut de France has quite recently elected Dr. von Kármán as Membre Correspondant de l'Académie. Being sure that you are all proud to learn of this distinction from one of the oldest scientific academies in the world, I am glad, as a French scientist, to pay on this occasion my tribute of admiration to Dr. von Kármán, in whom we all recognize one of the greatest among aerodynamicists, of whom he just gave such a nice definition."

Robert T. Jones, Aeronautical Engineer, Ames Aeronautical Laboratory, National Advisory Committee for Aeronautics: "Dr. von Kármán's 1935 Volta Congress paper has been studied intensively in recent years, and, needless to say, the new extensions and generalizations of the theory will be received with the greatest interest.

"It is believed that more positive statements can be made regarding the practical consequences of the sweepback effect. The tendency toward vanishing wave drag in the case of a long wing swept behind the Mach cone is illustrated by the result of the calculations. This effect, together with the design principles emphasized in the latter part of the lecture, should enable us to extend efficient flight speeds well into the supersonic range.

"Another effect of sweepback which may not be so widely appreciated is the effect on viscous flow. Following the Navier-Stokes equations, the oblique viscous flow pattern, when viewed in sections normal to the long axis of the wing, is seen to depend on the Reynolds Number and angle of attack derived from the component velocity. These considerations lead not only to the possibility of greater areas of laminar flow over the sweptback wing but, unfortunately, also to very low values of the usable maximum lift coefficient.

"Of the greatest interest to me, and I imagine to many others, is the theory for supersonic flow over wings of arbitrary plan form and section. Allen E. Puckett and I have recently been working on a similar problem. Although our methods appear more direct in certain cases, it seems that the new method may be better adapted to generalizations concerning mutual interference and overall drag.

"With regard to the new similarity rule for flows near the speed of sound, the question arises as to how far this rule depends on the assumed two-dimensional form of the flow. Insofar as the linear theory can be retained, it indicates an increasing effect of the aspect ratio, or the plan form, as the Mach Number approaches 1.0. Thus it appears that the three-dimensional flow around a sweptback wing near $M = 1.0$ is more appropriately described by two-dimensional patterns taken in planes at right angles to the flight direction than by two-dimensional flows taken in the usual sense. The extension of the similarity rule to three dimensions may therefore lead to some interesting results."

W. Bailey Oswald, Douglas Aircraft Company, Inc. (At the West Coast Presentation, Royce Hall of U.C.L.A., January 22, 1947): "Some years ago I was given an assignment to act as Chairman of the Aerodynamics Session at the Annual Meeting of the Institute of the Aeronautical Sciences in New York. At that time a great amount of experience had been obtained in flying aircraft at speeds corresponding to low Mach Numbers, that is, in the 'incompressible' region of speeds. However, airplanes were beginning to reach speeds where Mach Number effects could no longer be neglected, and it was evident that future progress depended upon knowledge of these compressibility effects. Consequently, I requested Dr. von Kármán, my former professor at the California Institute of Technology, to present a paper at the 1941 Annual Meeting on the subject, 'Compressibility Effects in Aerodynamics.' The paper, which he generously presented, is well known to all aeronautical engineers as an important contribution to higher speed flight.

"Although all the problems of flight at low and high subsonic speeds have by no means been solved, it is evident that the next major development in the progress of aeronautics will involve flight at supersonic speeds. The present comprehensive paper by Dr. von Kármán on the fundamental principles and applications of supersonic aerodynamics comes at a most opportune time. Again he has aided in clearing the way for future progress in aerodynamics.

"As an aerodynamicist who has been associated directly with the design of aircraft for many years, there are several thoughts suggested by the paper on which I should like to comment briefly.

"(1) Dr. von Kármán has pointed out three fundamental rules of supersonic aerodynamics. These are of great value to those concerned with supersonic flow. It might be considered that the area-velocity relation is also sufficiently basic so that the following rule could be added to those given: Streamlines in supersonic flows diverge as speed of flow increases and converge as speed decreases, and if the flow is started at subsonic speeds, the transition to supersonic speeds is made through a minimum section. This area-velocity relationship is the reverse of that experienced in subsonic flows.

"(2) The amount of mathematical knowledge in the supersonic region is in many respects equivalent to that in the subsonic region. This is particularly true regarding pressure distributions which in turn affect lift, moment, and drag. Experimental results agree with theoretical calculations at least as well as the designer has previously learned to expect. There is much in the science of subsonic flight which is yet analytically unsolved, but this situation has not prevented the design of efficient airplanes. Most test results showing the customary polar and stability diagrams of supersonic aircraft could well be mistaken for similar characteristics of subsonic designs. Considering the sound fundamental background in supersonic aerodynamics as shown in the paper, the agreement between theory and experiment, and the relatively normal action of bodies and surfaces, practical and efficient flight at supersonic speeds should soon be accomplished by the designers. However, this will require the same expenditure of effort now put into the development of subsonic types.

"(3) It is important to emphasize that the thickness-chord ratio appears squared in all the wave drag relations at supersonic speeds. At a value of this ratio of 4% the wave drag of a wing at a Mach Number of 2 approximately equals the skin friction, increasing very rapidly at higher thicknesses. The structural designers, who in the past have borne up so nobly under pressure from aerodynamicists for wings with more span and thinner sections, will be forced to the depths of despair. Likewise the demand for higher thrusts and lower fuel consumptions will be more insistent than ever before, since the drag and lift-drag ratio can hardly be expected to approach those customary in subsonic flight. Much progress which in the past has usually been credited to the aerodynamicist should be credited as well to the designers of the structure and power plant. In fact, it is likely that the quality of the airplane that could have been produced thirty years ago with today's power plants and structures would have been surprisingly high. The aerodynamicist will have even greater need of the aid of efficient structures and engines in the design of a supersonic airplane.

"(4) An unwritten rule which has been of value to the designer in the past might be termed the rule of external simplicity. Generally successful airplanes in the subsonic speed range have been simple and smooth externally (even at the expense of internal complexity). Furthermore they have been of generally 'conventional' straight forward conception. This rule might be well to bear in mind in our thinking of designs for supersonic flight.

"(5) Of great importance is knowledge of skin friction, the action of the boundary layer, its interaction

with shock waves, and their mutual effects on separation. It appears evident that any successful supersonic airplane cannot tolerate a large wave drag, so that skin friction will still comprise a large portion of the drag. Likewise separation of any magnitude cannot be tolerated. Therefore theoretical and experimental data on laminar and turbulent boundary layer effects at various Reynolds Numbers throughout transonic and supersonic speeds are greatly needed.

"(6) Perhaps beyond the scope of the paper but worth noting are the problems of flight near and through the transonic region where Mach Number equals nearly 1 and pressure waves progress at nearly flight speed. Unfortunately we must start from rest and go through these speeds before reaching supersonic conditions. The problems in supersonic aerodynamics are better understood than those in transonic aerodynamics. Some theoretical and experimental work in this region has been done, but the knowledge is still most inadequate. Certain devices, such as the use of sweep, delta wings, and boundary layer control, might be helpful in avoiding trouble, but valid theoretical and experimental results are necessary for safe controlled flight through transonic to supersonic speeds. Eventually flight through the sonic speed will become a matter of incidental importance with proper knowledge of proportioning the airplane.

"(7) The problems of designing some types of supersonic missiles and airplanes are likely to be quite different. Often the missile can be designed almost exclusively for the supersonic region since it can be boosted rapidly to high speed. In designing supersonic aircraft which are conventional in take-off and landing, the problems are multiplied by the necessity for constructing a single vehicle which will behave satisfactorily in three regions of flight: subsonic, transonic, and supersonic. It remains to be seen how serious the penalties will be in the final coordination of all factors involved in producing a suitable design, particularly considering modern standards of stability and control.

"Dr. von Kármán has summarized the principles and applications of supersonic aerodynamics by showing various ways of accomplishing supersonic flight either directly, or indirectly by use of sweepback and delta wings, and by demonstrating that the basic characteristics of lift, drag, and moment either have been or can be calculated for these configurations. The remaining problems are largely those of design, experience, and production. Today we have more basic knowledge of the aerodynamics of supersonic flight than that available in subsonic flight to the pioneers in aviation.

"When we have accomplished supersonic flight and are ready to consider the next regime, be that hypersonics, superaerodynamics, space-aerodynamics, or the like, we should again be most fortunate to have Dr. von Kármán present another of his authoritative papers in the new field."

Letters to the Editor

Dear Sir:

Mr. Flax's discussion (The Bending of Rotor Blades, JOURNAL OF THE AERONAUTICAL SCIENCES, Vol. 14, No. 1, January, 1947) of the various ways now available for evaluating the bending moments in rotor blades I found very interesting; but, since he states that my "type solution" method of *R. & M. 1875* (1939) is "not adaptable to the practical case of a blade of varying section," it seems probable that a paper I prepared for the British Association meeting of 1939 and which showed how to apply the method to varying sections cannot be well known in America. This paper was published in the British journal, *Engineering*, of 3 November 1939.

An inverse approach of the "type solution" kind described will enable the accuracy of any solution that has been obtained to be checked quite rapidly.

J. B. B. OWEN
Farnborough, Hants
England

Dear Sir:

The results obtained by E. Arthur Bonney, which appear in the February, 1947, issue of the JOURNAL, may be comprised in general, as well as simple, forms by the elementary procedure described below:

The upper boundary of the cross section of the parabolic biconvex wing of chord c and thickness ratio τ is given by

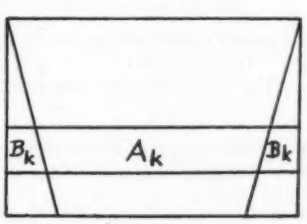
$$y = 2\tau[x - (x^2/c)], \; o \leq x \leq c \qquad (1)$$

If the trapezoid determined by the equally spaced ordinates $x_k = kc/n$, $k = 1, 2, \ldots, n$, is inscribed in the parabolic arch, Eq. (1), one finds, for the inclinations of the chords,

$$\beta_k = \tan \beta_k = \frac{y_k - y_{k-1}}{x_k - x_{k-1}} = \frac{2\tau}{n}(n - 2k + 1)$$

The lift for a typical area $A_k + 2B_k$ is then found to be

$$\frac{C_1\alpha}{n^2RB}\left(nRB - k + \frac{1}{2}\right)\left(2 + 4C_2\frac{n - 2k + 1}{n}\tau\right)$$



If such terms are summed from $k = 1$ to $k = n$, there results

$$\frac{C_L}{\alpha} = \frac{2}{3RB^2}\left(2C_3\tau + 6RB - 3 - \frac{2C_3\tau}{n^2}\right) \tag{2}$$

By similar procedure,

$$\frac{x_{c.p.}}{c} = \frac{(2C_3\tau - 2C_3\tau RB + 3RB - 2)n^2 + 2C_3\tau RB - 2C_3\tau}{(2C_3\tau + 6RB - 3)n^2 - 2C_3\tau} \tag{3}$$

If n is odd, the thickness ratio t/c of the polygonal section is $(n^2 - 1)\tau/n^2$. Hence, in the application of Eqs. (2) and (3), the value of τ is t/c or $n^2t/(n^2 - 1)c$, according as n is even or odd.

It is of course unnecessary to introduce a parameter to take the place of the wedge angle. The required characteristics, for all the cases under discussion, are given by Eqs. (2) and (3). The values for the double wedge, modified double wedge, and parabolic biconvex correspond, respectively, to $n = 2, 3$, and ∞. The values for a flat plate are obtained by placing $\tau = 0$. For a flat plate of infinite aspect ratio ($\tau = 0$, $R = \infty$),

$$C_L/\alpha = 4/B, \qquad x_{c.p.}/c = 1/2$$

The values of the characteristics (2) and (3), when n is odd (e.g., modified double wedge), are independent of n. In order to show this, one substitutes $\tau = n^2t/(n^2 - 1)c$ in Eqs. (2) and (3) and obtains

$$\frac{C_L}{\alpha} = \frac{2}{3RB^2}\left[2C_3\left(\frac{t}{c}\right) + 6RB - 3\right] \tag{4}$$

$$\frac{x_{c.p.}}{c} = \frac{3RB - 2 + 2C_3(t/c) - 2C_3(t/c)RB}{6RB - 3 + 2C_3(t/c)} \tag{5}$$

If now $n \to \infty$, then $t/c \to \tau$. Hence, the following theorem:

The values of C_L/α and $x_{c.p.}/c$ for a symmetric wing of polygonal section having an odd number of sides above the chord are given by Eqs. (4) and (5). These are equal to the respective values for the parabolic biconvex wing having the same thickness ratio.

One may, by partial differentiation of the general Eqs. (2) and (3), examine the variation of these characteristics with respect to τ, R, and B. From Eq. (2),

$$\frac{\partial}{\partial\tau}\left(\frac{C_L}{\alpha}\right) = \frac{4C_3}{3RB^2}\,\frac{n^2 - 1}{n} > 0$$

so that C_L/α increases with thickness ratio.

$$\frac{\partial}{\partial R}\left(\frac{C_L}{\alpha}\right) = \frac{2}{3R^2B^2}\left(3 - 2C_3\tau\,\frac{n^2 - 1}{n^2}\right)$$

and this quantity is positive if $C_3\tau < 3n^2/2(n^2 - 1)$. Now in the range of wedge angles and free stream Mach Numbers for which the theory applies, $C_3 < 5$ and $\tau < 0.3$. Hence, C_L/α increases with aspect ratio.

$$\frac{\partial}{\partial B}\left(\frac{C_L}{\alpha}\right) = \frac{4}{RB^6}\left[B^3 - RB^4 - \frac{2\tau(n^2 - 1)}{5n^2}(B^4 + B^3 + 5)\right]$$

This quantity is negative for all τ provided $RB > 1$, a requirement ordinarily fulfilled in practice, so that, subject to this condition, C_L/α decreases with increasing free stream velocity. In any case, the sign of the derivative is easily determined.

$$\frac{\partial}{\partial\tau}\left(\frac{x_{c.p.}}{c}\right) = \frac{2C_3n^2(n^2 - 1) - (6R^2B^2 + 6RB - 1)}{[2C_3(n^2 - 1)\tau + (6RB - 3)n^2]^2}$$

The sign of this quantity is that of the quadratic in RB appearing in the numerator. Hence, if $6RB > 3 + \sqrt{3}$, which again is normally the case, then $x_{c.p.}/c$ decreases with increasing thickness ratio.

$$\frac{\partial}{\partial R}\left(\frac{x_{c.p.}}{c}\right) = \frac{B(8C_3^2n^2\tau^2 - 4C_3^2n^4\tau^2 + 3n^4 - 4C_3^2\tau^2)}{(6Bn^2R + 2C_3n^2\tau - 3n^2 - 2C_3\tau)^2}$$

This quantity is positive if $C_3\tau < (\sqrt{3}/2)[(n^2/(n^2 - 1)]$. This condition is obviously fulfilled in the admissible range of values of C_3 and τ. Hence, $x_{c.p.}/c$ increases with aspect ratio.

The expression for $(\partial/\partial B)(x_{c.p.}/c)$ is clumsy, and its sign is that of a polynomial of degree 8 in B. It is perhaps most satisfactorily examined for particular values of τ, n, and R. Even with this approach, the calculations are tedious and will be omitted here. The quantity $x_{c.p.}/c$ does not, in general, vary monotonically with increasing free stream velocity. In the case of the flat plate, however, $x_{c.p.}/c$ increases with M.

The area bounded by the wing chord and the chords inscribed in the parabola is

$$\left(\frac{c^2}{3}\right)\left(\frac{n^2 - 1}{n^2}\right)\tau = f\left(\frac{n^2 - 1}{n^2}\,\tau\right)$$

On the other hand, $x_{c.p.}/c$ may be written in the form

$$\frac{x_{c.p.}}{c} = \frac{2C_3(1 - RB)\,\dfrac{n^2 - 1}{n^2}\,\tau + 3RB - 2}{2C_3\,\dfrac{n^2 - 1}{n^2}\,\tau + 6RB - 3} = F\left(R, B, \frac{n^2 - 1}{n^2}\,\tau\right)$$

Therefore wings of equal cross-sectional area have the same center of pressure location. The same statement applies to C_L/α.

EDWARD R. C. MILES
Research Engineer
The Glenn L. Martin Company

Dear Sir:

We were very interested in the paper by Messrs. Schubauer and Skramstad in the February issue of the JOURNAL OF THE AERONAUTICAL SCIENCES describing experiments in which a laminar flow was overlaid by a simple harmonic motion of the air in a wind tunnel, since we made rather similar experiments in 1930 on what we called "synthetic turbulence." The work was described in the *Physikalische Zeitschrift* (Vol. 32, p. 509) and later by one of us at a Symposium on Turbulence held by the Royal Society (but not printed in the proceedings) from which we quote the following:

"The experimental method which this line of thought suggests is to impose on a steady stream an S.H.M. of small amplitude and moderate frequency and to examine the transformation of the flow by this superposed fluctuation to see how closely it approximates to a naturally turbulent stream, particularly in respect of (1) rise and decay of the fluctuations and (2) development of characteristic distribution of velocity across the stream. The devices which have been employed to produce the desired S.H.M. in the course of the fluid comprise (1) tangential oscillation of part of a smooth wall over which the fluid passes, (2) passage of the fluid over a rigid corrugated solid boundary, (3) passage over a flexible boundary.

"Most illuminating from the point of view of the architecture of turbulence are experiments of the first type in which a pipe has a loose section interposed between the entrance cone and the main portion of the pipe into which air is aspirated. The loose section is mounted on the end of a long connecting-rod to a crank which is given a small throw by an electric motor. Air is first aspirated through the pipe at a speed just below the critical value and the gradient of mean velocity explored at various sections of the pipe downstream from the mobile portion, both with and without the latter in oscillation. Whereas the former profile was parabolic the latter goes over into the steeper type characteristic of turbulent flow, which can in fact be reproduced at a larger scale by increasing the speed of aspiration and leaving the loose section still. Fig. 1 shows two such profiles together with the amplitudes of velocity fluctuation opposite the oscillating

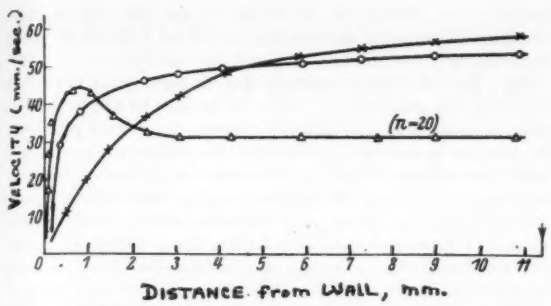


FIG. 1. Velocity distribution across pipe. Δ alternating flow; *x* laminar flow; *o* combined flow.

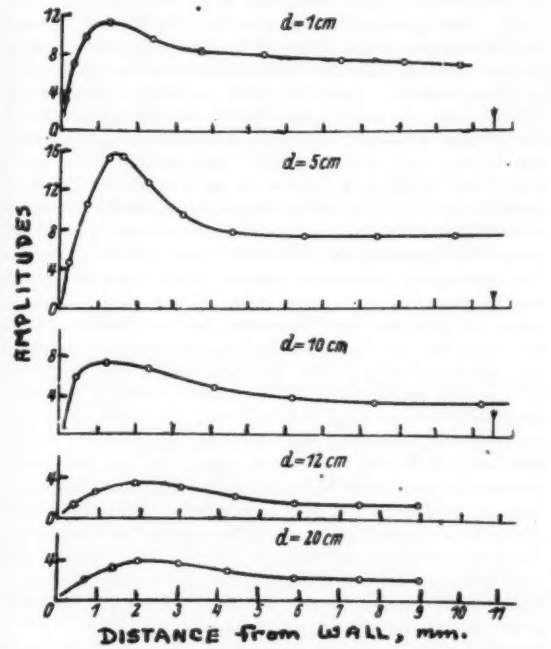


FIG. 2. Amplitudes of pulsations across pipe at various distances from origin.

section in the absence of a through draught. Fig. 2 shows profiles of the amplitude of the velocity fluctuations at an impressed frequency of 20 vibrations per sec. in various traverses distant *d* downstream from the mobile section while the air was being sucked into the pipe. The peak in the fluctuation appears to diminish in size as the air passes into the tube but admixture of smoke showed that the unsteadiness in the flow was being diffused into the core of the tube. It must be pointed out that the detecting apparatus (hot-wire) was made of purpose selective to the impressed frequency and takes no account of its possible modulation during the course of the pulsation through the pipe.

"It may be objected that this artificial turbulence does not correspond with nature in that the pulsation is not monochromatic in real turbulence but is more like a continuous spectrum. To the extent that such partial vibrations may in practice have different damping coefficients, so that the spectrum changes *en marchant*, our idealised picture of turbulent flow requires modification; but it is certainly a fact that natural turbulence shows velocity profiles and distribution of intensity of fluctuations precisely similar to that in our developed flow. Thus Fage and Townend (*Proc. Roy. Soc.*, Vol. 135A, p. 456) in their ultramicroscopic investigation of turbulent flow in pipes have found peaks in the fluctuations near the walls like those shown in our Fig. 2. One naturally asks whether or not there exists a critical frequency or wave-length of fluctuation which is most favourable to the metamorphosis here discussed. In our own pipe measurements with monotonal fluctuation, we found that increasing the frequency from 5 to 25 vibrations per sec. while keeping the amplitude constant caused a progressively fuller development of the velocity profile of turbulent flow, but as this particular observation has been made at one section we cannot say anything with regard to the relative rates of damping of these fluctuations."

We are pleased to note that Schubauer and Skramstad are investigating the rates of damping in question and look forward to further work by them on this subject. We should like to draw their attention to a point in connection with the use of the hot wire in close proximity to the wall of the tunnel. Because of the conduction of heat from the wire to the wall, readings of the hot wire, whether in respect of mean velocity or of velocity amplitude, are there in error since they require correction for this heat loss. We should, for instance, expect the location of 0.02 in. from the surface—one of those used—to fall within this error zone and so to require correction of the hot-wire readings, if this has not already been done.

E. G. RICHARDSON
E. TYLER
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Dear Sir:

Professor Eryk Kosko presented in his paper, "A Simplified Analysis of Hollow Beams with Longitudinal Webs Subjected to Torsional Loads" (JOURNAL OF THE AERONAUTICAL SCIENCES, Vol. 12, No. 1, p. 103, January, 1945), a method that led to a letter from Prof. Alfred S. Niles to the Editor (JOURNAL OF THE AERONAUTICAL SCIENCES, Vol. 12, No. 2, p. 240, April, 1945), in which he criticized Professor Kosko. The discussion concerning the method of Professor Kosko was closed by the answer of Professor Kosko and by an interesting Letter to the Editor from Leon Beskin (JOURNAL OF THE AERONAUTICAL SCIENCES, Vol. 12, No. 4, p. 407, October, 1945). But still one question was not solved satisfactorily. Namely, the one of the so-called "least work method." The writer was not in the position at the time when the discussion was taking place to express an opinion on this question, since it was necessary first to show that the existing proof of the so-called "least work method" is not correct. Since the question of any energy method is always of vital interest from a practical standpoint, the writer hopes that his remarks are not too late. In a recent paper the writer presented a proof showing that the definition of this method, cited in some textbooks, is incorrect (M. Z. Krzywoblocki, "On the So-Called Principle of Least Work Method," *Journal of the Franklin Institute*, Vol. 243, No. 3, p. 187, March, 1947).

In his letter Prof. A. S. Niles wrote among others: ". . . . as if the Maxwell-Mohr method were a less respectable substitute for a more fundamental method of minimum strain energy or 'least work.' Castigliano developed the method of least work by showing that the deflections required in the continuity equations could be obtained by differentiation of the strain energy of the loaded structure. . . . ." The following facts must be stated:

(a) Castigliano did not develop any method.

(b) Castigliano presented and proved certain theorems.

(c) To be completely valid, a method must be rigorosuly proved. There does not exist today any rigorous proof of the

so-called "least work" method. Consequently there does not exist any "least work" method.

(d) Application of Castigliano's theorem of Least Work is used successfully in certain cases but often in many other cases it is abused.

(e) The writer succeeded in finding in some textbooks on the subject of theory of elasticity, applied elasticity, theory of structure and strength of materials an attempt to prove the so-called "least work" method.

In a book on the subject of applied elasticity we read:

> "In considering structures with superfluous reactive elements it will always be assumed that the superfluous fastenings are removed and that corresponding superfluous reactive elements X, Y, Z are applied to the structure. If the superfluous fastenings are such that there are no displacements corresponding to the forces X, Y, Z, then from Castigliano's theorem it follows that $\partial V/\partial X = \partial V/\partial Y = \partial V/\partial Z = 0$ (a). The number of these equations is equal to the number of superfluous reactive elements; all these unknown quantities can be found from equations (a). It is easy to show that the second derivative of V with respect to any external force always will be positive. Due to the fact that the function $V = aX^2 + bY^2 + cZ^2 + dXY + \ldots$ must be positive for all values of X, Y, Z, ..., the coefficients a, b, c, ... for X^2, Y^2, Z^2, ... must be positive; therefore the second derivatives of V with respect to X, Y, Z, ... must be positive. Hence the previous equations (a) represent conditions of minimum V."

It is known that the necessary and sufficient conditions for minimizing a function of many variables are much more complicated than those presented above. The fact that all the second partial derivatives are positive does not mean that there is a minimum. There is another factor to consider before it can be concluded that the first partial derivatives of a function or set of functions representing the total strain energy of a structure in terms of statically indeterminate values will give a minimum. Namely, there is no upper limit imposed upon the magnitude of V. But the minimum obtained from these calculations may be so far from the minimum in actual conditions that the results cannot be reliable.

(f) With the exception of some textbooks the writer did not succeed in finding the expression "least work method" in any leading literature of the world on the subject of elasticity, theory of structure, etc. There is a term "least work theorem."

(g) Castigliano's Least Work Theorem has been applied successfully for many years to solution of problems in such types of structures as frames, beams, rings, etc. Certain authors, perhaps guided by the known statement that no special attention should be confined to concentrated forces and that the distributed loadings can also be brought within the scope of theorems, expressing the notion of the elastic strain energy (see R. V. Southwell, *Theory of Elasticity*, Oxford, Clarendon Press, 1936), applied this theorem directly to structures with continuously distributed stresses, obtaining sometimes almost unbelievable results. (An attempt by the writer to use this way and the resulting discrepancy between the calculated and actual stress distribution compelled him to doubt its validity.)

(h) The following proposition may, at least, serve in slightly clarifying the question: In order to be able to apply successfully and infallibly Castigliano's theorem to specific problems, we have to be assured *a priori*, that the redundant generalized forces or unknown stress distributions are expressed by such functions or such sets of functions, among which there is one, and only one, function or set of functions, which gives a smaller value to V than any other, after having first satisfied the boundary conditions (i.e., existence and uniqueness of an absolute minimum must be assured). After being assured that such an absolute minimum does exist and is unique, we present the strain-energy as a function of the redundant generalized forces (T_1, T_2, ...) or of the unknown coefficients. Next, we express in an analytical form the statement that the strain-energy has a stationary value for small variations in T_1, T_2, ... by equating to zero the first partial derivatives of V with respect to T_1, T_2, ... T_n. This gives the correct solution. In practice, it means that application of any arbitrary function may give results so far from the true ones that they cannot even be considered as an approximation. Hence, in order to obtain approximately correct results, one has to apply in each case a particular function. The problem of how to find the proper function in each particular case has not yet been solved in a rigorous way. One has to depend on intelligent guesses or on one's experience. But one should show in each particular case that the chosen function or set of functions will give a stress distribution that is in the first acceptable approximation sufficiently close to the actual one. The above-given proposition explains clearly when the application of Castigliano's theorem may be successful. When we apply this procedure to the calculation of the redundant generalized forces that can be analytically expressed in one and only one way, the procedure is usually successful, and we may be justified to a certain extent to call it a method (although the writer doubts whether this name may be used). This case occurs when we apply the theorem to the calculation of redundant generalized forces in beams, frameworks, rings, etc. In these elastic systems the generalized forces are usually expressed not as functions of coordinates but as pure numbers. The strain energies due to these generalized forces can be expressed in a unique way. When we apply this procedure to the calculation of the redundant generalized forces or stresses that are expressed as certain functions of coordinates with unknown coefficients, without *a priori* assurance that among the applied functions or sets of functions there is one and only one which gives a least value to V, the procedure may not be successful. Hence, it seems rather doubtful whether in these cases the procedure may be considered even as an approximate method. This is a random-choice method or procedure.

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Column Characteristics of Sandwich Panels Having Honeycomb Cores

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The Glenn L. Martin Company

SUMMARY

The column characteristics of panels having metal faces and honeycomb cores are reviewed. Compressive behavior is shown to be determined by the elastic properties of the face metal and by the shear rigidity of the stabilizing medium. A theory of column behavior, which considers the shear rigidity of the core and the reduced elastic properties of the face material at stresses above its elastic limit, is developed and shown to be in accord with experimental data. Optimum panel construction—i.e., the ratio of face thickness to total thickness—which yields the maximum strength-weight ratio is discussed and related to a loading parameter that measures the intensity of loading on the structure. Comparison of the structural efficiency of honeycomb panels with all-metal construction shows the sandwich to be the less efficient simple column. It is pointed out, however, that this comparison ignores the advantage of the transverse stiffness of the sandwich which comes into play in square panels or in those that are relatively long parallel to the direction of loading and narrow transversely.

SYMBOLS

A = area of cross section of core, sq.in.
B = intercept on stress axis of straight line tangent to Euler curve, lbs. per sq.in. (see Fig. 6)
b = panel width, in.
C = fixity coefficient
D = average specific gravity of sandwich
D_c = specific gravity of core
D_f = specific gravity of faces
d_t = deflection of flexure specimen, in.
E_f = modulus of elasticity of face material, lbs. per sq.in.
E' = effective modulus of elasticity of face material, lbs. per sq.in.
G_c = modulus of rigidity of core material, lbs. per sq.in.
k = $(D_f/D_c) - 1/2$
L = length of simply supported compression panel, in.
L_b = span of flexure specimen, in.
P = failing load of compression panel, lbs.
P_b = concentrated load on flexure specimen, lbs.
t = thickness of sandwich, center to center of faces, in.
t_c = thickness of core, in.
t_f = thickness of one face sheet, in.
u = $2t_f/t$
W = weight of sandwich, lbs. per sq.ft.
σ = stress in faces at failure, lbs. per sq.in.
ρ = radius of gyration of cross section of member, in.

INTRODUCTION

THE THEORETICAL ADVANTAGES of a material composed of stiff, dense faces separated and stabilized by a thick, light core have long been known, but only recently have actual combinations of surfaces and cores been investigated. A considerable interest in sandwich materials is attested by both technical publications and popular ones in this field.

It has been shown that, in general, the structural characteristics of a sandwich panel are primarily determined by the properties of both face and core material and by their relative proportions. The importance of excellent structural properties in the surface material is, of course, obvious and indicates the desirability of the light metals as face materials for sandwich constructions. The core must resist crushing and shearing from loads applied perpendicularly to the panel and must be stiff enough to prevent premature buckling of the surface sheets. The core need have no ability to resist loads applied in the plane of the panel, for these may be carried by the faces alone. The prime structural characteristics of a good core material are low specific gravity and a high ratio of shearing strength and stiffness to weight.

Many varieties of light materials have been investigated in recent years as potentially efficient core materials. Light woods such as balsa and quipo woods, porous plastics, expanded rubbers, and cellular constructions formed of tubes or honeycombs have all been considered. Each of these materials has demonstrated certain advantages and disadvantages, and each must be limited in its scope of application.

There is ample evidence to indicate that a relatively dense material formed into a honeycomb structure possesses distinct advantages as a stabilizing medium. This type of core, illustrated in Fig. 1, was first dis-

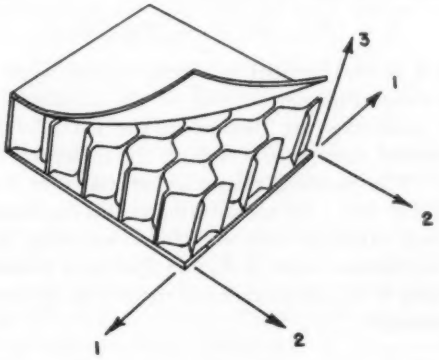

FIG. 1. Axis notation, honeycomb panel.

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closed as a stabilizing medium in sandwich constructions by Dornier,[1] but, until recently, there has been neither a practical method of producing a suitable honeycomb core nor a satisfactory adhesive for bonding metal faces to it.

Honeycomb constructions have been formed of various materials, such as impregnated paper, cotton fabric, and Fiberglas fabric, and have been made in various cell sizes. This paper is restricted to the discussion of the behavior as a column of sandwich constructions consisting of aluminum-alloy faces and a single type of plastic honeycomb core bonded with a special adhesive* developed by The Glenn L. Martin Company for this purpose. The data to be presented were obtained in the course of development work on this construction at this company. The core, chosen because of its general utility and economy, is described below.

Characteristics of Honeycomb Core

Material: Cotton fabric weighing 12 Gm. per sq.ft. impregnated with phenolic laminating varnish to a resin content of 50 per cent.

Geometry: Regular hexagonal cells; diameter of inscribed circle, 0.43 in.; wall thickness, 0.009 in.

Crushing Strength: 270 lbs. per sq.in. in 3-3 direction and calculated on gross cross-sectional area. (See Fig. 1 for axis notation.)

Compressive Modulus of Elasticity: 23,000 lbs. per sq.in. in 3-3 direction and calculated on gross cross-sectional area.

Modulus of Rigidity for Shear in the 1-2 plane parallel to the 1-1 axis and in the 2-3 plane parallel to the 3-3 axis: 8,200 lbs. per sq.in.

Shear Strength for shear in the 1-2 plane parallel to the 1-1 axis and in the 2-3 plane parallel to the 3-3 axis: 110 lbs. per sq.in.

Specific Gravity of Honeycomb Plus Adhesive: 0.08.

The modulus of rigidity is determined from flexural tests on simply supported specimens with concentrated load at mid-span. If the thickness of the faces is small with respect to the thickness of the sandwich, the deflection at mid-span is given by

$$d_t = (P_b L_b^3/48E_fI) + (P_b L_b/4bt_cG_c)$$

where I is the moment of inertia of the cover sheet areas about the neutral axis of the sandwich. The other quantities are defined in the list of symbols. The second term on the right is the deflection due to shear. The bending resistance of the core itself is practically zero. By substituting in this expression the observed values of load and deflection from the test and the known value of E_f and specimen dimensions, the value of G_c, the modulus of rigidity of the core, can be computed.

* This adhesive is marketed by the United States Plywood Corporation under the trade name "Plycozite."

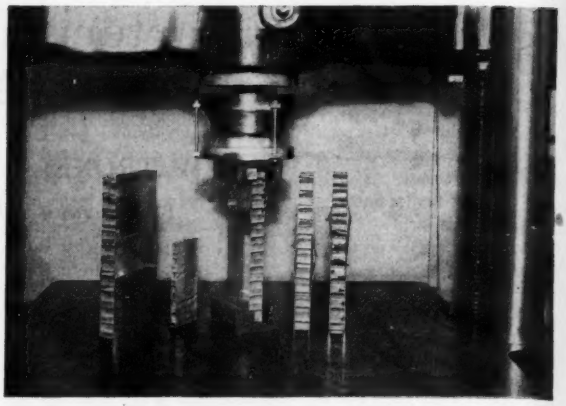

Fig. 2. Compression test setup and specimens.

Compressive Tests

Column tests reported in this paper were conducted on specimens having 24S-T aluminum-alloy Alclad or 75S-T Alclad faces bonded to the honeycomb core with a thermosetting adhesive developed especially for use in this construction. In all cases the 1-1 axis (see Fig. 1 for axis notation) of the honeycomb was parallel to the loading. In order to avoid premature failure at the column ends, clamps, as illustrated in Fig. 2, were employed. The spherical loading head was clamped during testing so as to prevent rotation. Test results are recorded in Table 1.

For panels having adequate bonds between faces and core, three primary types of compressive failure are possible: (1) by instability as a column (Euler-type failure); (2) by buckling of face sheets into the honeycomb cells; (3) by a short wave length wrinkling of the face sheets, usually referred to as a "wrinkling failure."

After compressive failure of a specimen, it is not always possible to determine by inspection the type of failure. In most cases one or both faces separates from the core, but this separation may be preceded by buckling of the faces into the honeycomb pattern. In some cases the separation is confined to a relatively short length; in this event the separated face sheet demonstrates a pronounced kink. Thicker faces tend toward complete separation from the core. Several different samples of compressive failures are illustrated in Fig. 2. There is generally no evidence that the bond between face and core is inadequate. Where faces separate from the core, cell walls are usually torn at the glue line, but the adhesive remains firmly affixed to the metal.

Later comparison of test results with theory indicates that wrinkling failures can occur only at impractically small slenderness ratios and that failure by local buckling limits the strength to short panels when the faces are thin. Cores with smaller cells, of course, permit higher failing stresses in the thinner covers.

TABLE 1
Column Test Data for Sandwich Panels Having Aluminum-Alloy Faces and Honeycomb Cores

Specimen	Overall Thickness (In.)	Column Length (L) (In.)	$\frac{L}{\rho\sqrt{C}}$	Width (In.)	Failing Load (Lbs.)	Stress in Faces at Failure (Lbs. per Sq.In.)
No. 2751	0.484	39.21	83.8	2.97	1,200	11,900
0.017 in.	0.488	29.97	63.5	2.97	1,925	19,050
24S-T faces	0.488	19.88	42.1	3.01	3,100	30,300
	0.491	9.9	20.5	2.99	3,600	35,400
	0.478	5.86	12.7	3.01	3,510	34,300
No. 2754	0.974	39.32	42.6	2.99	6,340	31,600
0.0335 in.	0.976	29.92	32.4	2.99	7,040	35,100
24S-T faces	0.976	19.93	21.6	2.97	7,920	39,800
	0.975	9.96	10.8	3.05	9,340	45,700
	0.976	6.02	6.5	3.02	9,040	44,600
No. 2765	0.519	28.45	56.6	3.03	2,225	22,250
0.0165 in.	0.524	10.33	20.0	2.98	3,500	35,500
24S-T faces	0.526	7.1	14.0	3.00	3,735	37,700
No. 2780	1.013	27.80	28.3	3.05	7,080	36,300
0.032 in.	1.016	17.86	18.2	3.10	8,340	40,300
24S-T faces	1.018	9.91	10.1	3.07	8,760	42,100
	1.007	5.99	4.6	3.01	8,900	46,300
No. 2753	0.947	39.4	42.3	2.97	3,125	31,900
0.0165 in.	0.948	30.0	32.2	2.93	3,415	35,300
24S-T faces	0.943	19.9	21.4	2.98	3,690	37,500
	0.946	9.93	10.7	2.98	3,910	39,700
	0.946	5.83	6.2	3.01	4,000	40,300
No. 2781	0.992	28.6	29.3	3.00	3,375	34,100
0.0165 in.	0.984	18.3	18.9	3.04	3,490	34,800
24S-T faces	0.981	9.85	8.5	3.03	3,650	39,800
	0.992	6.0	6.1	2.98	3,950	43,700
No. 2755	0.975	39.4	42	2.97	6,980	37,300
0.0315 in.	0.969	39.5	42	3.00	6,900	36,300
75S-T faces	0.970	29.9	31.8	2.99	9,760	51,800
	0.969	19.9	21.2	3.04	10,220	53,400
	0.971	9.6	10.2	3.00	10,750	56,800
	0.972	6.1	6.5	2.92	10,940	59,500
No. 2782	1.021	28.1	28.4	3.04	9,000	47,000
0.0315 in.	1.016	18.0	18.2	3.00	10,680	56,500
75S-T faces	1.015	10.0	10.1	3.00	10,770	57,100
	1.024	6.0	6.1	3.00	11,760	62,300

CALCULATION OF COLUMN STRENGTH

Any attempt to predict the compressive failing load of a sandwich panel must take into account the low shear stiffness of the core and also the reduction of the modulus of elasticity of the faces at stresses above the proportional limit. The column behavior of an ideally stabilized sandwich panel would be adequately described by the conventional short-column equation for the faces if the entire load is assumed to be carried by the faces alone. In actual sandwiches, however, this relation must be corrected for the lack of shear rigidity of the core.

Timoshenko[3] considers shear stiffness and expresses the buckling load of a column as

$$P_{cr} = P_e/[1 + (nP_e/AG)] \qquad (1)$$

where

P_e = Euler load = $\pi^2 EI/L^2$
A = cross-sectional area
G = modulus of rigidity
n = a factor depending on shape

For the sandwich construction, n is unity, $G = G_c$, and $A = bt_c$. However, it will simplify the calculation and introduce only a negligible error to take $A = bt$.

The resistance of the core to flexure is negligible; hence, $I = {}^1/_2 bt_f t^2$ and the radius of gyration equals ${}^1/_2 t$. Since the modulus of elasticity of the core is extremely small, the load carried by the core can be neglected. Hence, $P_{cr} = 2\sigma t_f b$, where σ is the stress in the face sheets at failure. Making these substitutions in Eq. (1) and solving for L,

$$L = \pi\rho\sqrt{(E/\sigma)[1 - (2t_f\sigma/tG_c)]} \qquad (2)$$

Let $2t_f/t = u$. Then,

$$L/\rho = \pi\sqrt{(E/\sigma)[1 - (u\sigma/G_c)]} \qquad (3)$$

From Eq. (3) the relation of slenderness ratio and failing stress can be computed for a pin-ended column. This can be generalized to cover other end conditions by substituting $L/\rho\sqrt{C}$ for the left-hand member, where C = the fixity coefficient.

For stresses above the proportional limit E is no longer constant but is a function of stress. The effective value of E is approximately equal to the tangent modulus derived from the compression stress-strain curve of the face material. However, for design purposes, it is more convenient to derive the effective E from the conventional straight-line column curve. It can readily be proved that, if B is the intercept on the stress axis of a straight line tangent to the Euler curve (see Fig. 3) and if E' denotes the effective modulus and E_f the modulus as usually defined,

$$E' = (27/4)[\sigma(B - \sigma)^2 E_f/B^3] \qquad (4)$$

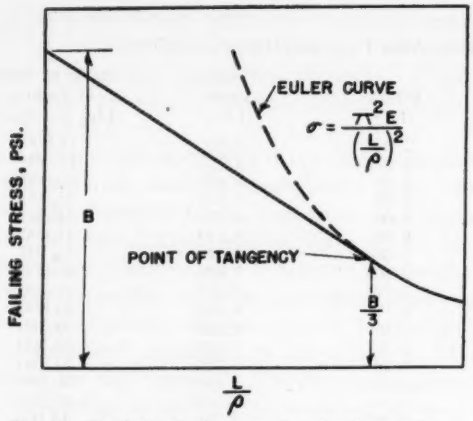

FIG. 3. Typical column curve for aluminum alloys.

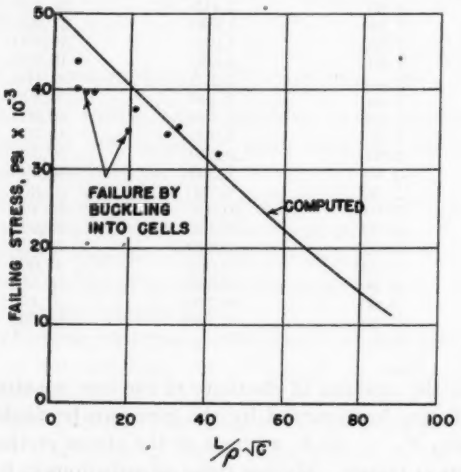

FIG. 4. Failing stress of columns; specimens 1 in. thick with 0.016-in. 24S-T Alclad faces.

This value of E', when substituted into the Euler formula, will give the straight-line column curve in the range $^1/_3B < \sigma < B$.

Hence, from Eqs. (3) and (4), for the short-column range

$$\frac{L}{\rho\sqrt{C}} = \frac{3\pi}{2}\left(1 - \frac{\sigma}{B}\right)\sqrt{\frac{3E_f}{B}\left(1 - \frac{u\sigma}{G_c}\right)} \quad (5)$$

And for the long-column range ($\sigma < {}^1/_3B$)

$$L/\rho\sqrt{C} = \pi\sqrt{(E_f/\sigma)[1 - (u\sigma/G_c)]} \quad (6)$$

Curves computed by means of Eq. (5) are compared with test results in Figs. 4, 5, and 6. In these computations the following values were used:

B = 50,000 lbs. per sq.in. for 24S-T Alclad
B = 80,000 lbs. per sq.in. for 75S-T Alclad
E_f = 9,500,000 lbs. per sq.in.
G_c = 8,000 lbs. per sq.in.

In plotting the test results, the effect of the clamps was neglected and C was assumed to be 4.0. The clamps

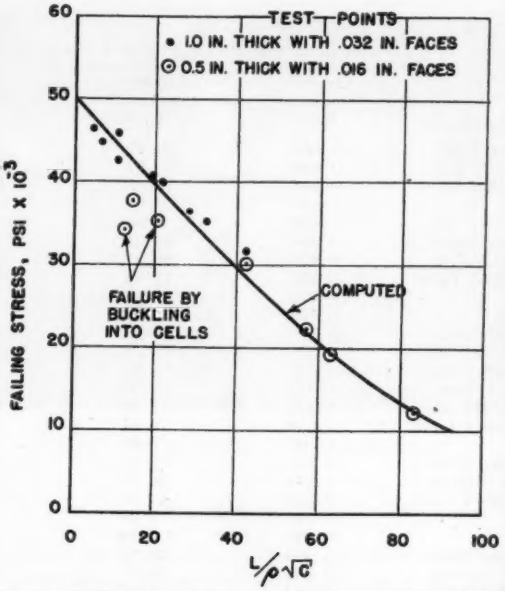
TEST POINTS
• 1.0 IN. THICK WITH .032 IN. FACES
⊙ 0.5 IN. THICK WITH .016 IN. FACES
FAILING STRESS, PSI X 10^3
FAILURE BY BUCKLING INTO CELLS
COMPUTED
$\frac{L}{\rho\sqrt{C}}$

FIG. 5. Failing stress of columns; specimens 0.5 in. thick with 0.016-in. 24S-T Alclad faces and 1.0 in. thick with 0.032-in. 24S-T Alclad faces.

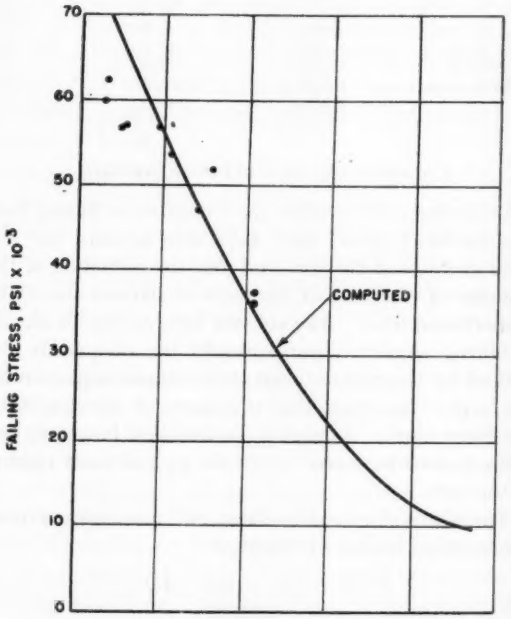

FIG. 6. Failing stress of columns; specimens 1 in. thick with 0.032-in. 75S-T Alclad faces.

undoubtedly tended to reduce somewhat the effective length of the column, but this effect is offset by assuming perfect fixity at the ends of the specimen; hence, the value of $L/\rho\sqrt{C}$ for the test specimens should be very nearly correct. The test points are all seen to lie either on or above the computed curve, except for the

short specimens where wrinkling or local buckling came into play.

Wrinkling Failure

The critical stress for wrinkling failure of an infinitely thick sandwich with an isotropic core is given by Gough, Elam, and deBruyne as follows:[4]

$$\sigma_c = [9E_c^2E_f/4(1 + \nu_0)^2(3 - \nu_0)^2]^{1/3} \qquad (7)$$

where

E_f = modulus of elasticity of face
E_c = modulus of elasticity of core
ν_0 = Poisson's ratio of core material

For cores of finite thickness the critical stress is approximately the same. Since the modulus of rigidity of an isotropic material = $E/2(1 + \nu)$, Eq. (7) can be expressed as

$$\sigma_c = [9G_c^2E_f/(3 - \nu_0)^2]^{1/3} \qquad (8)$$

where G_c = modulus of rigidity of core.

The honeycomb core is not isotropic, but an approximate estimate of the probable wrinkling stress can be made by using Eq. (8) and neglecting ν_0. Since the wrinkling stress is above the proportional limit, the effective modulus E' must be used in place of E_f. Hence, $\sigma_c = (G_c^2E')^{1/3}$ approximately. Using the relationship between stress and effective modulus of faces given by Eq. (4), the wrinkling stress for 24S-T bare material is computed to be 45,000 lbs. per sq.in., which corresponds to about 41,000 lbs. per sq.in. for Alclad. (The cladding contributes little to resistance to buckling at high stress; hence, the calculations for Alclad first neglect the cladding and then average the computed stress on the core metal over the entire thickness of the sheet.) Similar calculations for 75S-T Alclad indicate a wrinkling stress of approximately 55,000 lbs. per sq.in. Reference to the curves of Figs. 4 to 6 shows that the critical wrinkling stress can be reached only by panels of such small slenderness ratio as to be impractical and uneconomical.

Local Buckling

The results of a series of tests of short specimens with thin 24S-T Alclad faces is shown in Fig. 7. The failing stresses were variable, probably because of variation in the geometry of the honeycomb as prepared in the laboratory. Shown also in Fig. 7 is a curve of the estimated ultimate strength of the faces, computed on the assumption that the surface sheets are simply supported by a series of longitudinal walls spaced 0.43 in. apart. All except a few of the points lie above this line.

Optimum Thickness of Surface Sheets

In a sandwich construction the strength-weight ratio depends on the ratio of face sheet thickness to total thickness.[2] Adding core thickness increases stiffness and strength but also adds to weight. For any given case, there is one ratio of face thickness to total thickness which gives the maximum ratio of strength to weight. This is the optimum ratio. The optimum ratio depends on the intensity of loading of the panel, being greater for short heavily loaded panels than for long lightly loaded panels.

The efficiency criterion (or strength-weight ratio) for an axially loaded panel is the failing load per unit width divided by its weight per unit area. In this paper P/bW is used as the measure of efficiency, P/b being the load per inch width, and W the weight of the panel in pounds per square foot. For panels in compression, the value of the efficiency criterion depends on the length of the panel. A long panel will fail at a lower stress than a short one; hence, it is necessary to restrict comparison to panels of the same length and load carrying capacity. These factors are usually set by the design conditions, while the thickness and details of construction are left to the designer's choice.

Consider a set of geometrically similar pin-ended compression members made of the same material. In such a set, all members have the same slenderness ratio; hence, all fail at the same stress as shown by the column curve. The load carried by a member will be proportional to its cross-sectional area and, hence, proportional to (length)^2 or to length times width, since the members are geometrically similar. All members of this set fail at the same stress, and P/L^2 or P/bL has the same value for all.

For a panel supported at its ends only, the total ultimate load varies directly as the width. The load per unit width is therefore the significant quantity for comparisons, and for this reason P/bL is used as the

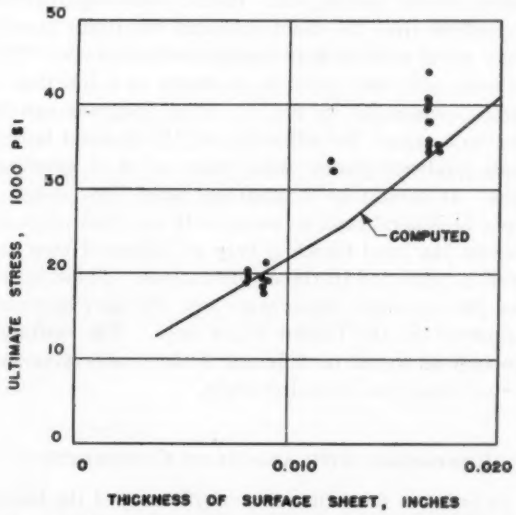

Fig. 7. Ultimate strength of 24S-T Alclad faces on honeycomb core.

measure of loading intensity for flat panels. It will be called the *loading parameter* for end-supported panels.

The method of comparing compression panels of various constructions is to plot the efficiency criterion P/bW as ordinate and loading parameter P/bL as abscissa. On such a plot, a series of geometrically similar panels should show as a single point. Comparison is made at the same value of loading parameter only; comparisons at different values are meaningless. The highest point at a given abscissa indicates the most efficient type of construction for the given intensity of loading (loading parameter).

For a long panel, simply supported along the edges parallel to the direction of loading, the failing load is independent of the length of the panel. The loading parameter for edge-supported panels is, therefore, taken as P/b^2. This measure of loading intensity may be derived by reasoning analogous to that used above for the case of the pin-ended column. Both derivations depend on the principle that the failing loads of geometrically similar structures of the same material are proportional to the square of their dimensions.

Eq. (2), which gives the relation of column length and failing stress, and Eq. (4) permit the optimum thickness ratio and the relation of efficiency criterion and loading parameter to be calculated for aluminum-alloy-faced sandwich panels supported at the ends. The method of calculation is outlined in Appendix 1. The optimum thickness ratio and optimum design stress are shown as functions of the loading parameter in Fig. 8. The curves indicate that short heavily loaded panels should be designed with greater relative thickness of covers and higher failing stress than long lightly loaded panels. These curves may be used for tentative selection of the most economical proportions of the sandwich, which must, of course, be modified for practical considerations such as availability of sheet gages, design details, lateral loading, etc. Rather wide departure of proportions from the exact optimum will result in relatively small reduction in strength-weight ratio. The optimum efficiency criterion is shown as a function of loading parameter in Fig. 9. This diagram can be used to compare the efficiency of Alclad-faced honeycomb sandwich panels with other types of constructions. It should be pointed out that Figs. 8 and 9 apply in general only to faces 0.016 in. thick or more, because the local buckling type of failure of thin material is neglected in these calculations. These figures give the optimum proportions and efficiency of panels supported at the loaded edges only. The optimum proportions would be different if the panels were supported along the unloaded edges.

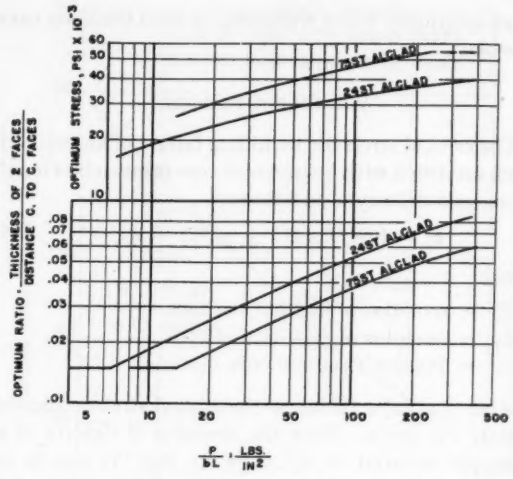

FIG. 8. Optimum stress and face thickness ratio for honeycomb sandwich panels.

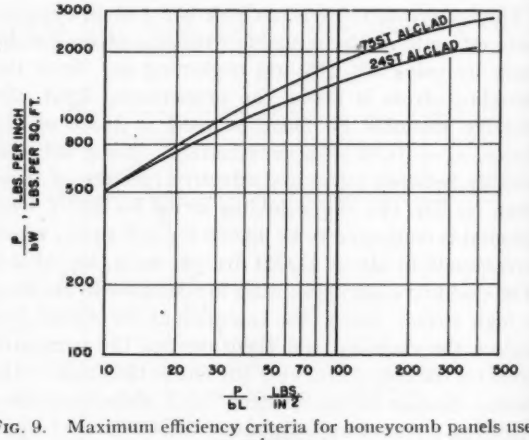

FIG. 9. Maximum efficiency criteria for honeycomb panels used as columns.

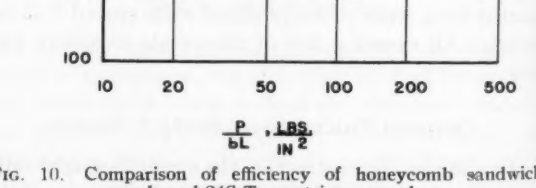

FIG. 10. Comparison of efficiency of honeycomb sandwich panels and 24S-T zee-stringer panels.

COMPARISON WITH ALL-METAL CONSTRUCTION

In order to show the relative efficiency of the honeycomb sandwich compared with all-metal construction, a curve of the optimum efficiency criterion for 24S-T

bare faces on honeycomb is shown in Fig. 10, with points computed from test results of 24S-T zee-stringer panels with flat skin reported by Schuette.[5] The tests on the zee-stringer panels are reported in terms of average failing stress, which can be converted to the efficiency criterion used here by multiplying by the constant factor 0.0695. It can be seen that the best zee panels are somewhat more efficient than the sandwich panels over the range of loading parameters covered by the tests and that the relative advantage of the former becomes smaller as the loading parameter decreases. This, however, is only a rough indication of the relative efficiency of the two types of construction, because only a complete comparison including weight of transverse supporting members, connections, and other details would serve to determine the lightest construction. Comparing panels supported only at the ends ignores the advantage possessed by the sandwich construction by reason of its transverse stiffness. This transverse stiffness makes the sandwich type particularly suited to panels of nearly square dimensions or panels that are relatively long parallel to the direction of loading and narrow transversely. Weight-saving due to elimination of supporting ribs or transverse formers is a large factor of the sandwich construction.

Another advantage of the sandwich is the smooth surface it presents, while the sheet-stringer combination may show a buckled surface at relatively low load. This is important for the exterior surface of high-speed aircraft.

Appendix 1.—Derivation of Optimum Face-Thickness Ratio

Given the length of a panel between supports and the compressive load per inch width, it is required to find the ratio of the face thickness to sandwich thickness which will give a panel of minimum weight. The panel is to be supported at its loaded edges only.

Neglecting the load carried by the core, which has negligible stiffness,

$$P = 2t_f b\sigma = utb\sigma \tag{1A}$$

The weight of the panel per square foot is

$$W = 5.2[D_f ut + D_c(t - {}^1/_2 ut)]$$

or

$$W = 5.2D_c t(1 + ku) \tag{2A}$$

Eliminating t from Eqs. (1A) and (2A),

$$P/bW = u\sigma/5.2D_c(1 + ku) \tag{3A}$$

From Eq. (2) the relation of length and failing stress is

$$L = \frac{\pi}{2} t \sqrt{\frac{E'}{\sigma}\left(1 - \frac{u\sigma}{G_c}\right)} \tag{4A}$$

Eliminating t from Eqs. (1A) and (4A),

$$\frac{P}{bL} = 2\sigma u \Big/ \left[\pi \sqrt{\frac{E'}{\sigma}\left(1 - \frac{u\sigma}{G_c}\right)}\right] \tag{5A}$$

In Eqs. (3A) and (5A), the efficiency criterion P/bW and the loading parameter P/bL may be regarded as functions of the independent variables u and σ. It is more convenient, however, to deal with their logarithms; hence, let $X = \log P/bL$ and $Y = \log P/bW$. Since P, b, and L are given quantities in this problem, P/bL is constant; hence, $dX = 0$. Optimum conditions obtain when $dY = 0$ and $dX = 0$ for arbitrary values of du and $d\sigma$. (The question of whether this condition gives a minimum or maximum for $P/b\,W$ must be investigated. For this problem it can be readily proved to be a maximum.) Hence,

$$(\partial X/\partial u)(\partial Y/\partial \sigma) - (\partial X/\partial \sigma)(\partial Y/\partial u) = 0 \tag{6A}$$

From Eqs. (6A), (4A), and (5A), remembering that E' is a function of σ,

$$ku^2\frac{\sigma}{G_c} - u\left[2k + \frac{\sigma}{G_c}\left(1 - \frac{\sigma}{E'}\frac{dE'}{d\sigma}\right)\right] + 1 - \frac{\sigma}{E'}\frac{dE'}{d\sigma} = 0 \tag{7A}$$

Eq. (7A) expresses the relation that must hold between u and σ to make the efficiency criterion a maximum at a given value of the loading parameter.

Let

$$1 - (\sigma/E')(dE'/d\sigma) = R$$

Substituting in Eq. (7A) and solving for u,

$$u = \frac{G_c}{\sigma} + \frac{R}{2k} - \sqrt{\left(\frac{G_c}{\sigma}\right)^2 + \left(\frac{R}{2k}\right)^2} \tag{8A}$$

This is the optimum value of u. The other root of the quadratic gives a value of $u > 2G_c/\sigma$, which would make L imaginary; hence, it has no physical meaning.

When $R/2k$ is small with respect to G_c/σ,

$$\text{Optimum } u = \frac{R}{2k}\left[1 - \frac{\sigma}{2G_c}\left(\frac{R}{2k}\right)\right] \text{approx.} \tag{9A}$$

From Eq. (4) and the definition of R, the following expression for R in the range ${}^1/_3 B < \sigma < B$ is derived:

$$R = 2\sigma/(B - \sigma) \tag{10A}$$

when $\sigma < {}^1/_3 B$, E' is constant; hence, $R = 1$.

The procedure in calculating the curves of Fig. 10 is as follows:

(1) Assume σ.

(2) Compute optimum u from Eqs. (10A) and (9A) or (8A).

(3) Using the assumed value of σ and optimum u, compute P/bL from Eq. (5A) and P/bW from Eq. (3A).

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[4] Gough, G. S., Elam, C. F., and deBruyne, N. A., *The Stabilization of a Thin Sheet by a Continuous Supporting Medium*, Journal of The Royal Aeronautical Society, Vol. 44, p. 12, January, 1940.

[5] Schuette, E. H., *Charts for the Minimum-Weight Design of 24S-T Aluminum Alloy Flat Compression Panels with Longitudinal Z-Section Stiffeners*, N.A.C.A. Report ARRL5F15, August, 1945.

Discussion

Dr. N. J. Hoff, Professor of Aeronautical Engineering, Polytechnic Institute of Brooklyn: "In predicting the buckling loads of sandwich-type columns, Mr. Troxell and Mr. Engel have used the Engesser formula, which takes into account both the bending and shearing deformations of the column. The formula is valid for columns that are weak in shear because of their material or the shape of their cross section. It is only approximately correct for sandwich columns that consist of two strong faces and a core that is weak in shear. For this latter type of column more accurate formulas were recently developed by investigators in England, Holland, and this country. The situation is characteristic of the duplication of work caused by wartime restrictions on communication. The English paper is entitled 'Flat Sandwich Panels Under Compressive End Loads.' It was written by D. Williams, D. M. A. Leggett, and H. G. Hopkins and was published as *AD Report No. 3174* of the Aeronautical Research Committee in June, 1941. The Dutch publication is '*Die Stabilität geschichteter Streifen*,' by A. van der Neut, and was published as *Report No. S284* of the Nationaal Luchtvaartlaboratorium in August, 1943. The American article was written jointly by Dr. S. E. Mautner and the writer and was presented at the Sixth International Congress for Applied Mechanics, September 22 to 29, 1946, in Paris, France, under the title 'Bending and Buckling of Sandwich Type Beams.' The results of these three papers are equivalent, and they differ little from the Engesser formula in the case where the thickness of the faces is small as compared to the thickness of the core. For this reason the Engesser formula appears to be entirely satisfactory for the calculation of the buckling load of the specimens tested at The Glenn L. Martin Company.

"In their weight comparison Messrs. Troxell and Engel found that flat sandwich panels were slightly less efficient than the best possible reinforced aluminum-alloy panels. On the other hand, the British investigators proved theoretically that curved sandwich panels were considerably more efficient in carrying compressive end loads than were conventional reinforced aluminum-alloy panels of airplane structures. These calculations are contained in *Report No. SME 3203* of the Royal Aircraft Establishment entitled 'Sandwich Panels and Cylinders Under Compressive End Loads,' written by D. M. A. Leggett and H. G. Hopkins, August, 1942. Consequently sandwich airplane structural elements can be expected to reach even better strength-to-weight ratios than was predicted by the authors in their objective evaluation of the results of investigations carried out by The Glenn L. Martin Company."

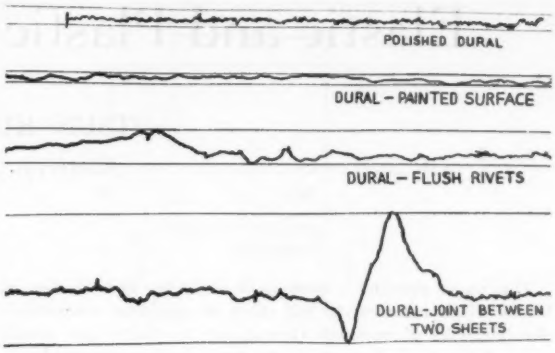
L. N. Phillips, British Commonwealth Scientific Office, United Kingdom Scientific Mission: "I think this is an extremely valuable and necessary piece of work. The current interest in sandwiches will remain so much talk unless adequate design data based on sound theory are forthcoming. I think this paper does exactly that.

"It is interesting to see that the almost classic paper by deBruyne, Gough, and Elam forms the primary reference to most papers on sandwiches. It may not be so generally known that deBruyne also had the germ of an idea for honeycombs. Some years ago, I think in 1939, he took out a patent for the production of honeycomb by molding sheets of material between solid hexagonal bars stacked in an orderly pile. After cure, the metal bars would be ejected from the mass so as to leave the outline of a grid in the sheet material. I think the authors of the paper have been modest in their comparison of the sandwich and stringer types of construction. Their remarks in the concluding paragraph deserve some amplification.

"There is tremendous interest on both sides of the Atlantic in high-speed aircraft, and the smoothness of the airfoil is a critical factor here. The stringer construction with the thin skin unsupported in many places allows considerable local deflection. I am speaking now in terms of 1 or 2 thousandths of an inch. The buckling pattern on an airfoil often becomes visible with loads as low as 60 per cent of the design load, and the waveness of the surface will affect the airflow before this point. In fact, the buckling pattern is often studied by playing powerful lights on the compression face at an acute angle so as to put the depressions into shadow and reflect the highlights into the lens of a camera placed directly above the surface. This technique reveals whether the load applied is being taken into the structure as evenly and efficiently as possible.

"There are perhaps two main schools of thought on the subject of getting smooth wing surfaces. The first method is to use a thick metal skin over an other-

wise normal stringer construction, so that the critical load for short wave-length buckling is never reached. I have seen some so-called smooth-wing sections made at great trouble and expense, and the main difficulties are: (1) The flush rivets are not really flush. (2) Because of slight differences in the edge thickness of sheet material and slightly uneven rib surface finish, the joints between sheets show a slight step. This is shown in the illustration herewith where the vertical scale is magnified. So far then we have no really practical production method for smooth-wing manufacture along these lines.



"With regard to the sandwich type of construction, the paper indicates that local buckling into the honeycomb cells is avoided by using dural greater than about 0.017 in. in thickness. The problem of rivets does not exist, because the surface sheets are glued and joints can be made flush on the surface by removing a few thousandths of an inch from the honeycomb below the projecting edge."

Omission

The following acknowledgments were omitted in the article "The Rate of Descent of Parachutes from Various Altitudes," by Kenneth E. Penrod, George L. Maison, and James E. McDonald (JOURNAL OF THE AERONAUTICAL SCIENCES, Vol. 14, No. 5, pp. 303–310, May, 1947):

(1) The experiments reported were conducted by the Wright Field Aero Medical Laboratory.
(2) Fig. 3 was published by courtesy of the Kollsman Instrument Division of the Square D Company.

Erratum

Correction to page 294, May, 1947, JOURNAL (Letter to the Editor, C. C. Lin): The last word in the second line of the second paragraph should be 1944 instead of 1946.

Notice

A Symposium on Applied Mathematics will be held at Brown University, Providence, R.I., August 2–4, 1947, under the sponsorship of the American Mathematical Society. The Institute of the Aeronautical Sciences, The American Society of Mechanical Engineers, and the American Institute of Physics will act as cosponsors. The subject of the Symposium will be "Non-Linear Problems in the Mechanics of Continua."

All persons interested in receiving further information concerning the Symposium should write to Prof. W. Prager, Brown University, Providence 12, R.I. Copies of the program and reservation cards will be sent out to them 3 weeks before the date of the Symposium.

Elastic and Plastic Bending of Beams*

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Northrop Aircraft, Inc.

SUMMARY

This paper presents a method of analyzing fiber stresses of beams which includes the full effect of nonlinear stress-strain characteristics of materials throughout the elastic and plastic range of bending. Various shaped cross sections commonly encountered in aircraft practices are considered. A new term "norstress" is proposed for the measurement of remote fiber stresses of cross sections of beams under pure bending. Eqs. (11) and (12) or Eqs. (14) and (15) are the final equations derived for calculating the norstress on the rectangular beams. Eqs. (17) and (18) are for □, [, I, Z, ⌐⌐, L, and T beams; Eqs. (21) and (22) for circular beams; and Eqs. (23) and (25) for tubular beams. A stress-strain formula is also recommended for use of calculations of integrals involved in the stress and moment equations, as well as for the purpose of drawing stress-strain curves in the absence of test data.

INTRODUCTION

THE CLASSIC THEORY of pure bending in which the bending stress is calculated by the formula My/I was developed on the basis of two assumptions: first, that plane cross sections of a beam before bending remain plane after bending, i.e., the elongations and contractions of longitudinal fibers are proportional to the distance from the neutral surface; second, that Hooke's law is valid, which means that there is a linear variation of normal stress distribution along the depth of the cross section of the beam varying from zero at the neutral axis to a maximum at the extreme fiber. These two assumptions combined reveal that this theory is good only when the stress-strain characteristic of the material concerned is linearly related.

For bending of beams of materials such as plain low carbon steel, the classic theory can be applied with reasonable accuracy when the stresses are within the elastic limit. However, when the extreme fiber stress exceeds the proportional limit of the materials, this theory can no longer be applied, because the stress distribution over the cross section is no longer linear but tends to follow the stress-strain characteristic of the material as shown in Fig. 1.

A number of aircraft materials have no proportional limit (or limit of proportionality) as a definite point. In a great many cases the proportional limit may exist but is low in comparison with the ultimate strength of materials. These facts render the ordinary flexure formula $f_b = Mc/I$ inapplicable.

It has been general aircraft practice to use "form factors" or "bending modulus of rupture" to take care of the difference between the amount of the actual stress in the material and the stress calculated by the conventional formula $f_b = My/I$. The main objections to the use of "form factor" or "bending modulus of rupture" can be stated as follows:

(1) The variation of strength of beams depends not only on the form of the cross section of the beams but also on the relative dimensions of this cross section. Therefore, a complete set of form factors would have to be made available for each material as well as for each combination of the shape and dimensions.

(2) It is time-consuming to calculate C/I for hollow, open, and tubular sections in practical application.

There have been a great many advancements to the pure bending theory of beams, in which the effect of nonlinear stress-strain characteristic of materials is taken into account. Saint-Venant,[1] Timoshenko,[2] Meyer,[3] von Kármán,[4] Herbert,[5] Bach,[6] and so on, have attacked this problem either analytically or graphically. These authors dealt principally with methods for determining the general solution of the problem.

Also, some authors[7] have expressed the stress-strain relationship mathematically by considering stress as a function of strain or vice versa, but the attempt has been limited by the mathematical difficulties encountered in applying it to specific problems. Some[8] have approximated this relationship by assuming the actual stress-strain distribution to be trapezoidal or rectangular in shape. However, this method failed to show accurate results.

To eliminate these difficulties, a consideration is made by furthering the development of the exact theory, as well as by using a bending stress distribution based on

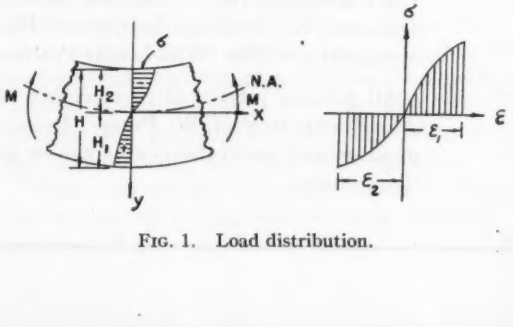

FIG. 1. Load distribution.

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* The author is very much indebted to D. E. Nelson for his invaluable assistance in the development of this paper. Thanks are also extended to R. C. Skinner, W. Hoffman, and R. A. Skinner, who worked on the calculations and drawings.

† Senior Structures Engineer. Now, Director of Aeronautical Engineering Department, National North-West University, Sian, Shensi, China.

the actual stress-strain characteristics of the materials. This consideration is extended to several cases of beams with different shapes of cross sections commonly used in aircraft practice.

To facilitate aircraft design of beams, curves of M/BH^2 against σ_t or h/H for rectangular box, channel, I, T, and L beams, and M/R^3 against σ_t or D/t for circular and tubular beams may be constructed by using the equations derived. The expression M/BH^2 or M/R^3 would seem more preferable than "bending modulus of rupture," "form factors," and so on, because it can be seen from the curves in this analysis that for design calculations it is easy to choose the dimensions of the beams desired. For convenience, M/BH^2 and M/R^3 will be used, and, therefore, are referred to as the "norstress*."

Symbols

A = area of the cross section of the beam
B = breadth of the beam
b = inner breadth of the hollow or open cross section
c = perpendicular distance from a principal axis to the extreme fiber
E = modulus of elasticity
e = 2.71828.....
H = depth of the beam
h = inner depth of the hollow or open cross section of the beam
I = moment of inertia
K = arbitrary constant
M = moment
R = outer radius of the circular cross section
r = inner radius of tubular beam
t = thickness of flanges of the beam
x, y = mutually perpendicular axes along and perpendicular to the neutral surface of the beam
ϵ = strain
σ = stress
γ = radius of curvature
$\log_e$ = natural logarithms

Subscripts

c or 2 = compression
t or 1 = tension
y = yielding
max. = maximum
ult. = ultimate

General Assumptions

(1) Material is homogeneous and isotropic.
(2) No local crippling occurs.
(3) No residual stresses exist.

Fundamental Consideration

The general theory regarding this problem is well known and a brief discussion is presented here merely for the sake of completeness.

* "Norstress" with a dimension equal to moment/length^2 or lbs. per sq.in. is a newly proposed term which represents a measurement of maximum normal stress (remote fiber stress) of cross sections of beams due to bending and was originated in the Northrop Stress Group by the present writer.

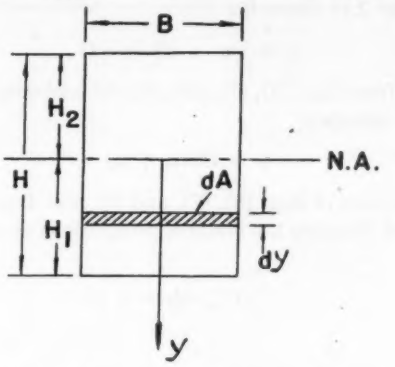

Fig. 2. Cross section of rectangular beams.

When a beam is subjected to pure bending, the total internal normal stresses over a cross section must be equal to zero according to the law of equilibrium. Also, these stresses represent a couple and the moment produced by this couple is equal to the moment applied. Analytically, these statements can be expressed respectively as follows:

$$\int_{-H_2}^{H_1} \sigma dA = 0 \tag{1}$$

$$\int_{-H_2}^{H_1} \sigma y dA = M \tag{2}$$

where σ denotes the normal stresses on the cross section of the beam concerned; M represents the moment applied; y indicates the distance from the neutral surface to the various fibers, and H_1, H_2 the distances to the most remote fibers; dA is a differential area of the cross section; ϵ_1 and ϵ_2 show the elongation and contraction of the most remote longitudinal fibers. Hereafter, the first equation will be referred to as the stress equation and the second as the moment equation. Even though materials of beams do not follow Hooke's law, experiments with beams show that, during bending, the elongations and contractions of longitudinal fibers are proportional to their distances from neutral surfaces of the beams. Therefore, the following relation[2] holds

$$\epsilon = (y/\gamma) \tag{3}$$

where γ is the radius of curvature of the beams.

Hence,

$$\epsilon_1 = H_1/\gamma, \qquad \epsilon_2 = H_2/\gamma$$

and

$$H_1/H_2 = \epsilon_1/\epsilon_2 \tag{4}$$

Also, the sum of H_1 and H_2 is equal to the total depth of the beam, that is

$$H_1 + H_2 = H \tag{5}$$

Calculations of Stresses of Rectangular Beams

It is easily seen that

$$dA = Bdy \tag{6}$$

(See Fig. 2.) From Eq. (3)

$$y = \gamma\epsilon, \qquad dy = \gamma d\epsilon \tag{7}$$

Again, from Eqs. (3), (4), and (5), the following relation can be obtained:

$$\gamma = H/(\epsilon_1 + \epsilon_2) \tag{8}$$

Substitution of Eqs. (6), (7), and (8) into Eqs. (1) and (2), and changing the limits of integrations accordingly, yield

$$\int_{-\epsilon_2}^{\epsilon_1} \sigma d\epsilon = 0 \tag{9}$$

and

$$1/(\epsilon_1 + \epsilon_2)^2 \int_{-\epsilon_2}^{\epsilon_1} \sigma\epsilon d\epsilon = M/BH^2 \tag{10}$$

Procedure for Calculating the Stress Distribution

In practical application, Eqs. (9) and (10) should be changed to somewhat different forms as follows:

$$\int_0^{\epsilon_1} \sigma d\epsilon = \int^{-\epsilon_2} \sigma d\epsilon \tag{11}$$

and

$$\frac{1}{(\epsilon_1 + \epsilon_2)^2}\left(\int_0^{\epsilon_1} \sigma\epsilon d\epsilon + \int_{-\epsilon_2}^{0} \sigma\epsilon d\epsilon\right) = \frac{M}{BH^2} \tag{12}$$

By observation of Eq. (11), it is apparent that for any assumed value of maximum elongation ϵ_1, there is a corresponding value of maximum contraction ϵ_2. Also, the position of the neutral axis of the cross section of the beam, hence the stresses and strains, not only depends on the shape of the stress-strain curve of the material, but also on the magnitude of the bending moment applied. If a pair of values of ϵ_1 and ϵ_2 obtained by Eq. (11) corresponds to the external moment M, the values of ϵ_1, ϵ_2, and M must satisfy Eq. (12). Therefore, the stress distribution can be obtained by solving Eqs. (11) and (12).

A. Graphical Method

(1) Stress-strain diagrams for both tension and compression must be prepared first. This can be done either by test data or by analytical method (see Appendix A)

(2) Assuming a series of values of maximum elongation ϵ_1, the corresponding values of maximum contraction ϵ_2 can be determined by Eq. (11) and item (1).

(3) Substitute each pair of corresponding values of ϵ_1 and ϵ_2 into Eq. (12). A curve with ϵ_1 as abscissas and M/BH^2 as ordinates can be plotted.

(4) For practical applications, it is more convenient to plot curves of σ_1 and σ_2 against M/BH^2 as shown in Figs. 3 than of ϵ_1 and ϵ_2 vs. M/BH^2. This can be done by using items (1) and (3).

(5) The neutral axis can be determined by equations

$$H_1 = H/[1 + (\epsilon_2/\epsilon_1)]$$
$$H_2 = H/[1 + (\epsilon_1/\epsilon_2)]$$

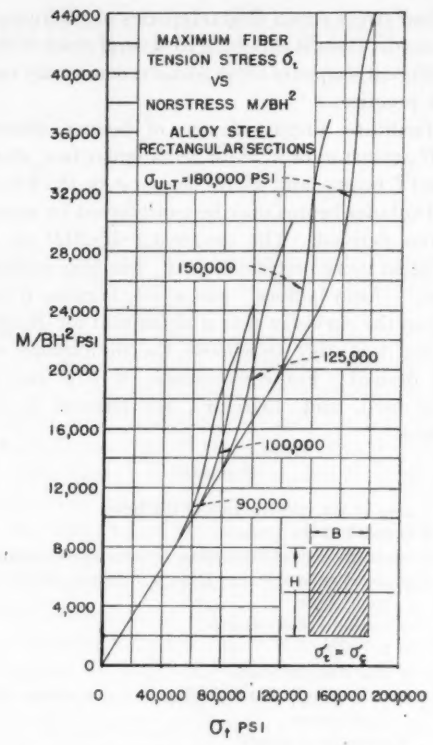

FIG. 3a. Norstress curves for rectangular beams—alloy steels.

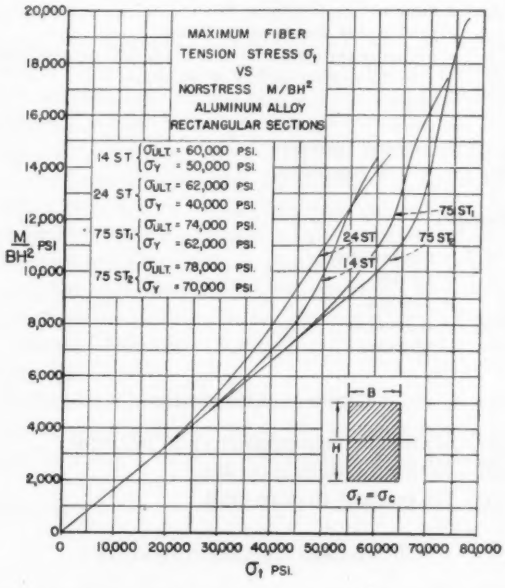
MAXIMUM FIBER TENSION STRESS σ_t VS NORSTRESS M/BH^2 ALUMINUM ALLOY RECTANGULAR SECTIONS
14 ST σ_{ULT} = 60,000 PSI. σ_Y = 50,000 PSI.
24 ST σ_{ULT} = 62,000 PSI. σ_Y = 40,000 PSI.
75 ST$_1$ σ_{ULT} = 74,000 PSI. σ_Y = 62,000 PSI.
75 ST$_2$ σ_{ULT} = 78,000 PSI. σ_Y = 70,000 PSI.
$\frac{M}{BH^2}$ PSI
σ_t PSI
B
H
$\sigma_t = \sigma_c$

FIG. 3b. Norstress curves for rectangular beams—aluminum alloys.

(6) The various fiber strains, hence stresses, may be calculated by

$$\epsilon_t = (\epsilon_1/H_1)y, \qquad \epsilon_c = (\epsilon_1/H_2)y$$

where subscripts t and c indicate tension and compression, respectively.

B. *Analytical Method*

For this purpose, a stress-strain formula is recommended as follows:

$$\epsilon = (\sigma/E) + \epsilon_{max.} e^{K[(\sigma/\sigma_{ult.})-1]} \tag{13}$$

where

$$K = \log_e(0.002/\epsilon_{max.})/[1 - (\sigma_y/\sigma_{ult.})]$$

is constant for each particular material.

By observing the following integrations by parts:

$$\int \sigma d\epsilon = \sigma\epsilon - \int \epsilon d\sigma \tag{a}$$

$$\int \sigma\epsilon d\epsilon = (\sigma\epsilon^2/2) - {}^1/_2 \int \epsilon^2 d\sigma \tag{b}$$

and substituting Eq. (13) into Eqs. (11) and (12), the following equations are obtained. [See Eqs. (g) and (h) in Appendix B.]

$$\frac{\sigma_1^2}{2E} + \epsilon_{1\,max.}\left(\sigma_1 - \frac{\sigma_{1\,ult.}}{K_1}\right)e^{K_1[(\sigma_1/\sigma_{1\,ult.})-1]} + \frac{\epsilon_{1\,max.}\sigma_{1\,ult.}}{K_1 e^{K_1}} =$$

$$\frac{\sigma_2^2}{2E} + \epsilon_{2\,max.}\left(\sigma_2 - \frac{\sigma_{2\,ult.}}{K_2}\right)e^{K_2[(\sigma_2/\sigma_{2\,ult.})-1]} + \frac{\epsilon_{2\,max.}\sigma_{2\,ult.}}{K_2 e^{K_2}} \tag{14}$$

$$\frac{1}{(\epsilon_1+\epsilon_2)^2}\left[\frac{\sigma_1^3}{3E} + \frac{\epsilon_{1\,max.}}{EK_1^2}(K_1^2\sigma_1^2 - K_1\sigma_1\sigma_{1\,ult.} + \sigma_{1\,ult.}^2) \times \right.$$

$$e^{K_1[(\sigma_1/\sigma_{1\,ult.})-1]} + \frac{\epsilon_{1\,max.}^2}{4K_1}(2K_1\sigma_1 - \sigma_{1\,ult.})e^{2K_1[(\sigma_1/\sigma_{1\,ult.})-1]} +$$

$$\frac{\epsilon_{1\,max.}\sigma_{1\,ult.}}{K_1^2 e^{K_1}}\left(\frac{K_1\epsilon_{1\,max.}}{4e^{K_1}} - \frac{\sigma_{1\,ult.}}{E}\right) + \frac{\sigma_2^3}{3E} + \frac{\epsilon_{2\,max.}}{EK_2^2}(K_2^2\sigma_2^2 -$$

$$K_2\sigma_2\sigma_{2\,ult.} + \sigma_{2\,ult.}^2)\, e^{K_2[(\sigma_2/\sigma_{1\,ult.})-1]} + \frac{\epsilon_{2\,max.}^2}{4K_2}(2K_2\sigma_2 -$$

$$\left.\sigma_{2\,ult.})e^{2K_2[(\sigma_2/\sigma_{1\,ult.})-1]} + \frac{\epsilon_{2\,max.}\sigma_{2\,ult.}}{K_2^2 e^{K_2}}\left(\frac{K_2\epsilon_{2\,max.}}{4e^{K_2}} - \frac{\sigma_{2\,ult.}}{E}\right)\right] =$$

$$\frac{M}{BH^2} \tag{15}$$

Where subscript 1 indicates tension characteristics of the material and subscript 2 compression. These two equations can be used to calculate the stress distribution of the beam by a procedure similar to that used in the graphical method.

Calculation of Stresses of Beams with Box, Channel, I, Z, ⊓, T, or L Sections

The method previously used in investigating beams with rectangular cross sections can be applied to this case without any difficulties. The equilibrium equation of internal forces over a cross section of the beam which corresponds to Eq. (1) will be

$$\int_{-H_1}^{H_1} \sigma dA - \int_{-h_1}^{h_1} \sigma dA = 0 \tag{16}$$

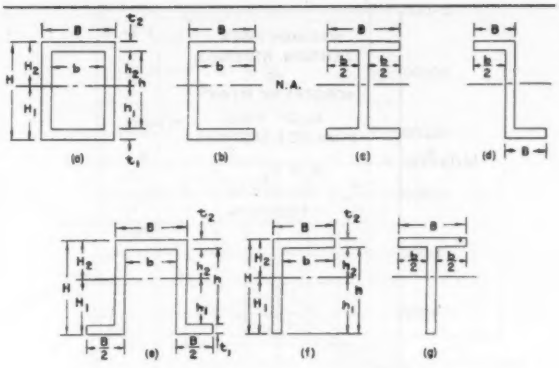

FIG. 4. Box, channel, I, Z, ⊓, T, and L sections.

where h_1 and h_2 are the distances from the neutral axis to the inner sides of the flanges of the sections concerned, as shown in Fig. 4. By using the same procedure for Eq. (11), the following equation can be obtained:

$$\int_0^{\epsilon_1} \sigma d\epsilon - \frac{b}{B}\int_0^{\epsilon_1'} \sigma d\epsilon = \int_0^{-\epsilon_2} \sigma d\epsilon - \frac{b}{B}\int_0^{-\epsilon_2'} \sigma d\epsilon \tag{17}$$

where

$$\epsilon_1' = (h_1/H_1)\epsilon_1$$

or

$$\epsilon_1' = \epsilon_1 - [t(\epsilon_1+\epsilon_2)/H]$$

and

$$\epsilon_2' = (h_2/H_2)\epsilon_2$$

or

$$\epsilon_2' = \epsilon_2 - [t_2(\epsilon_1+\epsilon_2)/H]$$

in which the strains ϵ_1' and ϵ_2' correspond to h_1 and h_2, respectively. In the same manner, the moment equation that corresponds to Eq. (12) can be derived as follows:

$$\frac{1}{(\epsilon_1+\epsilon_2)^2}\left[\int_0^{\epsilon_1} \sigma\epsilon d\epsilon + \int_{-\epsilon_2}^{0} \sigma\epsilon d\epsilon - \right.$$

$$\left.\frac{b}{B}\left(\int_0^{\epsilon_1'} \sigma\epsilon d\epsilon + \int_{-\epsilon_2'}^{0} \sigma\epsilon d\epsilon\right)\right] = \frac{M}{BH^2} \tag{18}$$

In Fig. 4 it is seen that if t_2 vanishes, the channel section becomes an angle section and the I section becomes a T section, as shown in Eqs. (f) and (g).

Using Eqs. (17) and (18), curves of M/BH^2 vs. σ_t (Figs. 5) are drawn for steel and aluminum-alloy beams with symmetrical sections. These curves are drawn with ratios of $b/B = 0.6, 0.8, 0.9, 0.98$, and $h/H = 0.6, 0.8, 0.9, 0.98$. In Figs. 6, curves with h/H as abscissas and M/BH^2 as ordinates for various ultimate and yielding stresses are plotted.

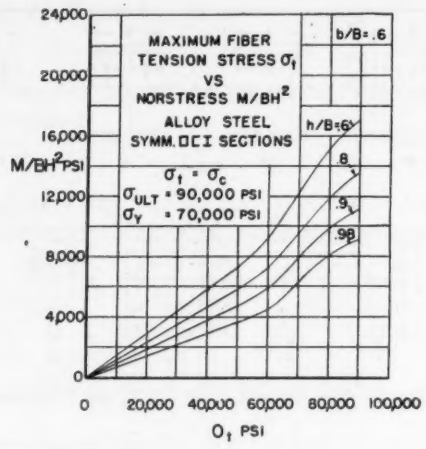

FIG. 5a. Norstress curves for □, ⊏, and I sections.

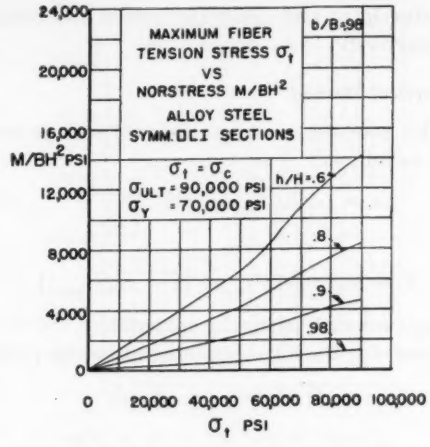

FIG. 5d. Norstress curves for □, ⊏, and I sections.

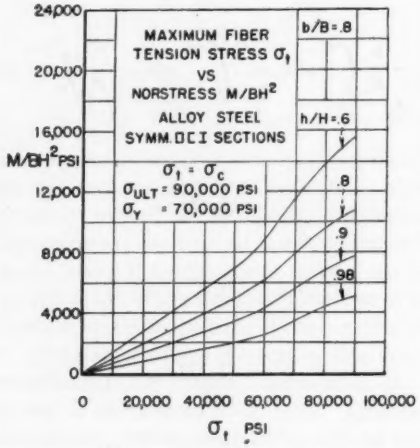

FIG. 5b. Norstress curves for □, ⊏, and I sections.

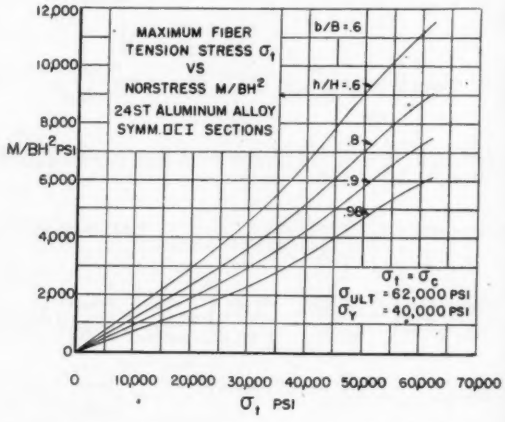

FIG. 5e. Norstress curves for □, ⊏, and I sections.

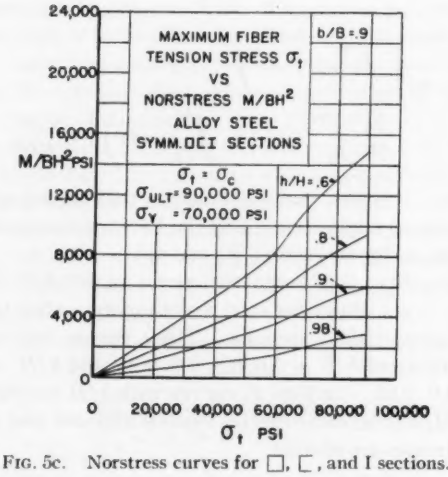

FIG. 5c. Norstress curves for □, ⊏, and I sections.

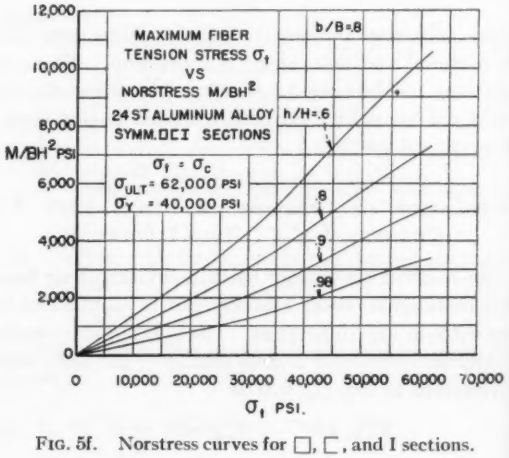

FIG. 5f. Norstress curves for □, ⊏, and I sections.

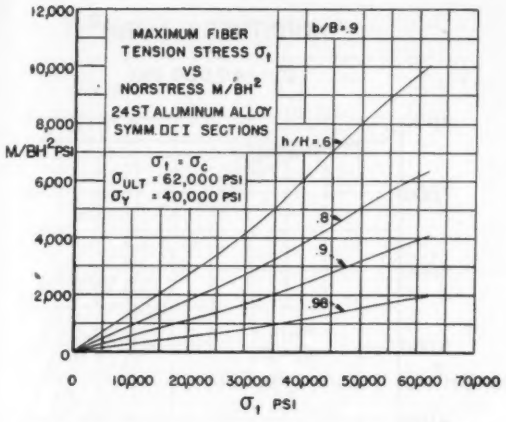

FIG. 5g. Norstress curves for □, ⊏, and I sections.

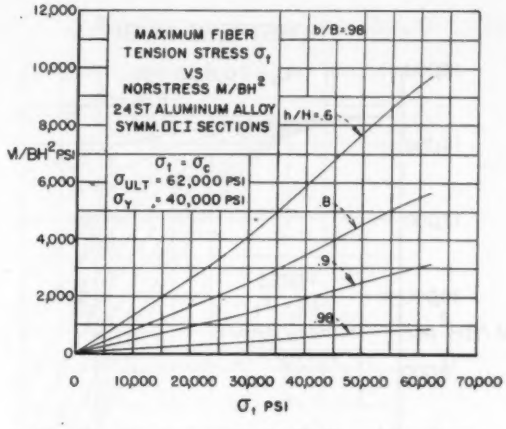

FIG. 5h. Norstress curves for □, ⊏, and I sections.

CALCULATIONS OF STRESSES OF CIRCULAR BEAMS

The previous consideration can also be applied to circular beams. (See Fig. 7.) In this case, the width B is no longer a constant but varies with y as related below

$$B = 2\sqrt{R^2 - (y + R - H_1)^2}$$

Hence,

$$dA = 2\sqrt{R^2 - (y + R - H_1)^2}\, dy \qquad (19)$$

Eqs. (3), (4), (5), (7), and (8) still hold for this case except that H is replaced by $2R$. Substitution of them into Eq. (19) yields

$$dA = \frac{4R^2}{\epsilon_1 + \epsilon_2}\sqrt{1 - \left(\frac{2}{\epsilon_1 + \epsilon_2}\epsilon + \frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + \epsilon_2}\right)^2}\, d\epsilon \qquad (20)$$

With the aid of Eqs. (3), (4), (5), (7), (8), and (20), Eq. (1) becomes

$$\int_0^{\epsilon_1} \sigma\sqrt{1 - \left(\frac{2}{\epsilon_1 + \epsilon_2}\epsilon + \frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + \epsilon_2}\right)^2}\, d\epsilon = \int_0^{-\epsilon_2} \sigma\sqrt{1 - \left(\frac{2}{\epsilon_1 + \epsilon_2}\epsilon + \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\right)^2}\, d\epsilon \qquad (21)$$

Similarly, a corresponding moment equation can be obtained by deriving from Eq. (2)

$$\frac{8}{(\epsilon_1 + \epsilon_2)^2}\left[\int_0^{\epsilon_1} \sigma\epsilon\sqrt{1 - \left(\frac{2}{\epsilon_1 + \epsilon_2}\epsilon + \frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + \epsilon_2}\right)^2}\, d\epsilon + \int_{-\epsilon_2}^{0} \sigma\epsilon\sqrt{1 - \left(\frac{2}{\epsilon_1 + \epsilon_2}\epsilon + \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\right)^2}\, d\epsilon\right] = \frac{M}{R^3} \qquad (22)$$

By using the same procedure, design curves with σ_1 or σ_2 as abscissas and M/R^3 as ordinates can be plotted.

CALCULATIONS OF STRESSES OF TUBULAR BEAMS

The technique used in the last two cases can be adopted to calculate the stress distribution for this case. (See Fig. 8.) The stress equations can be expressed as follows:

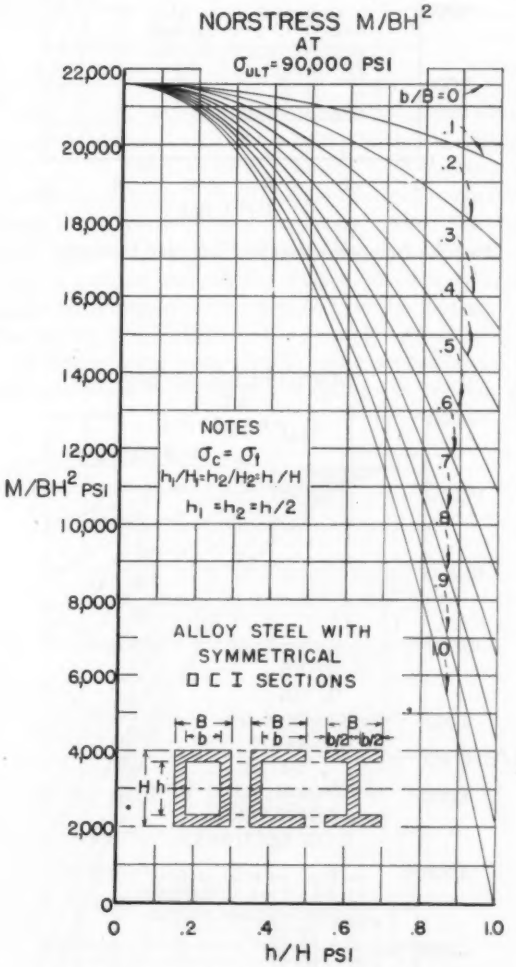

FIG. 6a. Norstress curves for □, ⊏, and I sections.

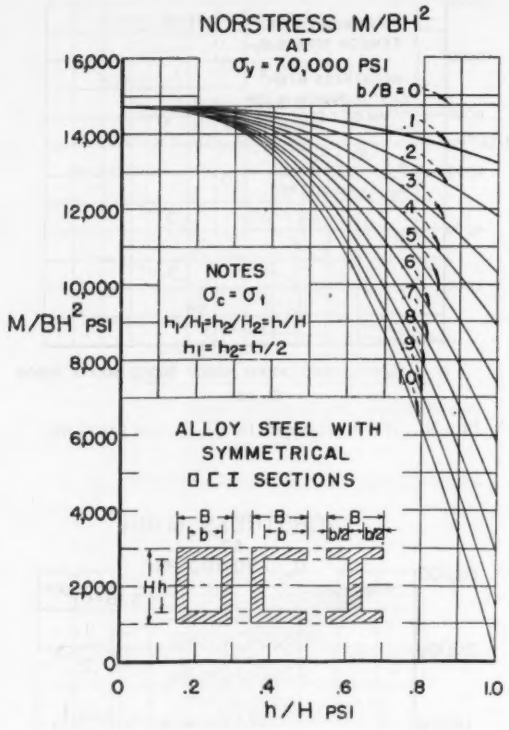

FIG. 6b. Norstress curves for □, C, and I sections.

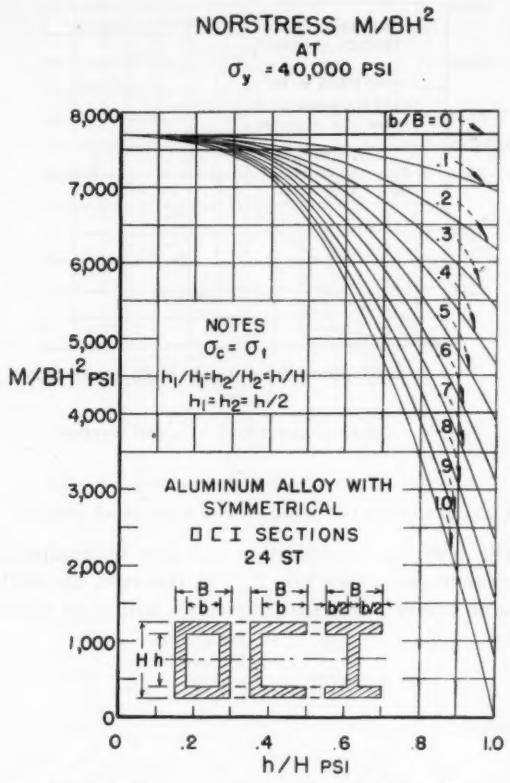

FIG. 6d. Norstress curves for □, C, and I sections.

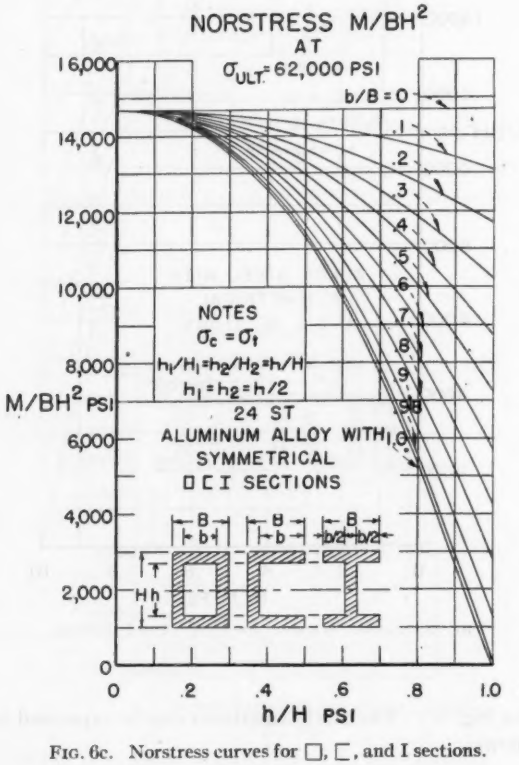

FIG. 6c. Norstress curves for □, C, and I sections.

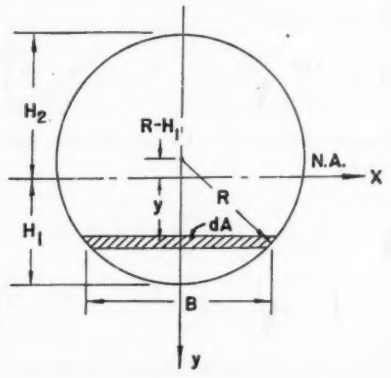

FIG. 7. Cross section of circular beam.

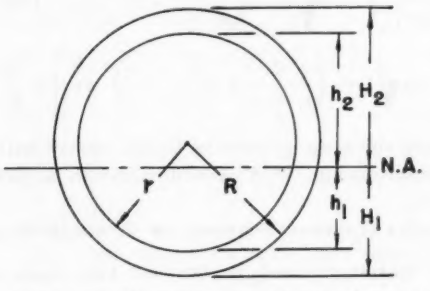

FIG. 8. Cross section of tubular beam.

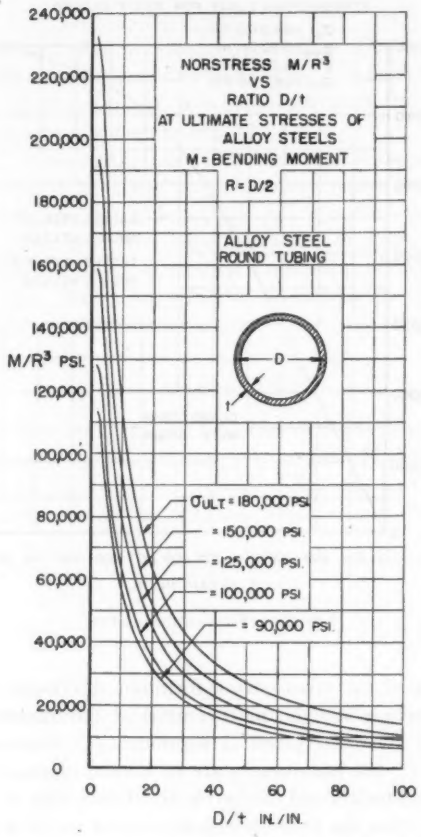

FIG. 9a. Norstress curves for circular and tubular beams.

$$\frac{R^2}{\epsilon_1+\epsilon_2}\int_0^{\epsilon_1}\sigma\sqrt{1-\left(\frac{2}{\epsilon_1+\epsilon_2}\epsilon+\frac{\epsilon_2-\epsilon_1}{\epsilon_1+\epsilon_2}\right)^2}d\epsilon-$$

$$\frac{r^2}{\epsilon_1'+\epsilon_2'}\int_0^{\epsilon_1'}\sigma\sqrt{1-\left(\frac{2}{\epsilon_1'+\epsilon_2'}\epsilon+\frac{\epsilon_2'-\epsilon_1'}{\epsilon_1'+\epsilon_2'}\right)^2}d\epsilon=$$

$$\frac{R^2}{\epsilon_1+\epsilon_2}\int_0^{-\epsilon_2}\sigma\sqrt{1-\left(\frac{2}{\epsilon_1+\epsilon_2}\epsilon+\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\right)^2}d\epsilon-$$

$$\frac{r^2}{\epsilon_1'+\epsilon_2'}\int_0^{-\epsilon_2''}\sigma\sqrt{1-\left(\frac{2}{\epsilon_1'+\epsilon_2'}\epsilon+\frac{\epsilon_1'-\epsilon_2'}{\epsilon_1'+\epsilon_2'}\right)^2}d\epsilon \quad (23)$$

where r is the inner radius of the tubular beam and the other symbols have the same meanings as in the previous cases.

Since

$$\epsilon_1' = (h_1/H_1)\epsilon_1, \qquad h_1 = H_1 - (R - r)$$

and

$$H_1 = 2\epsilon_1 R/(\epsilon_1 + \epsilon_2)$$

Therefore,

$$\epsilon_1' = {}^1/_2[(\epsilon_1 - \epsilon_2) + (r/R)(\epsilon_1 + \epsilon_2)]$$

Similarly

$$\epsilon_2' = {}^1/_2[(\epsilon_2 - \epsilon_1) + (r/R)(\epsilon_1 + \epsilon_2)] \quad (24)$$

The moment equation for tubular beams is

$$\frac{8}{(\epsilon_1+\epsilon_2)^2}\left[\int_0^{\epsilon_1}\sigma\epsilon\sqrt{1-\left(\frac{2}{\epsilon_1+\epsilon_2}\epsilon+\frac{\epsilon_2-\epsilon_1}{\epsilon_1+\epsilon_2}\right)^2}d\epsilon+\right.$$

$$\int_{-\epsilon_2}^{0}\sigma\epsilon\sqrt{1-\left(\frac{2}{\epsilon_1+\epsilon_2}\epsilon+\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\right)^2}d\epsilon-$$

$$\int_0^{\epsilon_1'}\sigma\epsilon\sqrt{\frac{r^2}{R^2}-\left(\frac{2}{\epsilon_1+\epsilon_2}\epsilon+\frac{\epsilon_2-\epsilon_1}{\epsilon_1+\epsilon_2}\right)^2}d\epsilon-$$

$$\left.\int_{-\epsilon_2'}^{0}\sigma\epsilon\sqrt{\frac{r^2}{R^2}-\left(\frac{2}{\epsilon_1+\epsilon_2}\epsilon+\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\right)^2}d\epsilon\right]=\frac{M}{R^3} \quad (25)$$

A family of curves of M/R^3 against D/t with various values of $\sigma_{ult.}$ can be drawn by using Eqs. (23) and (25) for routine stress calculations. Examples are given in Figs. 9.

CONCLUSION

Eqs. (11), (12), (17), and (18) are developed for rectangular beams as well as for beams with box, channel, I, Z, ⊓, T, and L sections. In practical applications, curves of norstress M/BH^2 against the most remote fiber stress σ or ratio h/H may be drawn. Also, Eqs. (21), (22), (23), and (25) are useful for beams with cross sections of circular and tubular shapes. In this case, norstress M/R^3 against the ratio D/t can be plotted. All the curves thus constructed are simple to use.

If beams have symmetrical cross sections and the characteristics of stress-strain curves in tension and

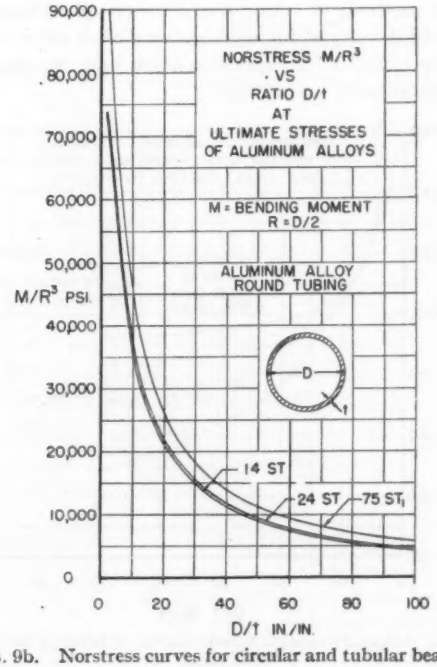

FIG. 9b. Norstress curves for circular and tubular beams.

compression are identical, the derived formula can be simplified a great deal, and, consequently, the amount of work for plotting the norstress curves will be greatly reduced.

By using Eq. (13) stress-strain curves are drawn, which are, in turn, used for plotting norstress curves. In doing this, an average value of maximum elongation equal to 0.06 is assumed for steel and 24 ST, 0.035 for 14 ST, and 0.040 for 75 ST.

In calculating the bending modulus of rupture, the general formula Mc/I is used, and in calculating the norstress, the formula M/BH^2 or M/R^3 is developed. Comparing I/c with BH^2 or R^3, it can be seen that for any shape of cross section, especially for hollow, open and tubular sections, application of the norstress curves to design results in considerable timesaving. Therefore, the norstress curves would seem preferable for general design purposes.

A comparison between the present curve and some results previously published by other authors for round chrome molybdenum steel tubing heat-treated to 150,000 lbs. per sq.in. is shown in Fig. 10. It indicates that there are some discrepancies between them. The fact that Cozzone's curve is higher than that of the author may be the result of his approximate assumptions for stress distributions over the cross sections of beams under bending. The differences between the ANC-5 curve and the author's for the values of D/t larger than 13 possibly results from the effects of local crippling and ovalization. From a survey of the tests in the past, it shows that the ANC-5 curves are also conservative. However, it should be borne in mind that channel, box, I, Z, ⊓, etc., sections are also subject to local crippling when allowable loads are computed. Although methods of computing local crippling have not been well developed for such sections when subjected to bending loads, a fair approximation may be obtained with the present knowledge.

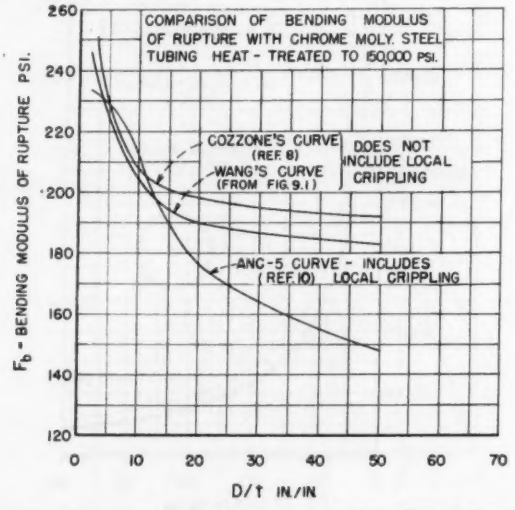

FIG. 10. Comparison with various curves of bending modulus of rupture.

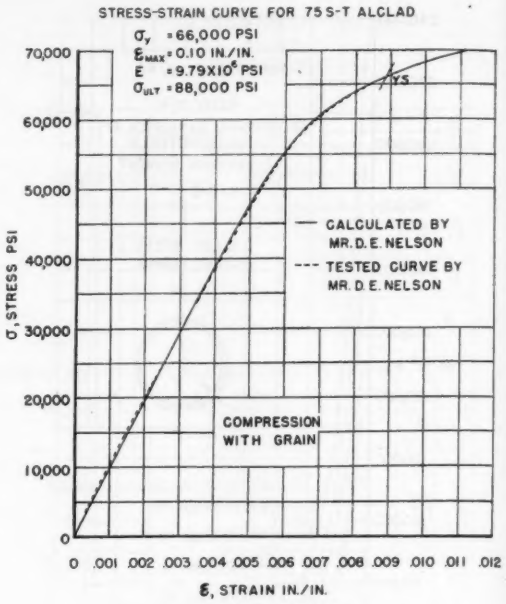

FIG. 11. Stress-strain curve.

Most of the stress-strain formulas developed in the past include one or more empirical parameters that usually have no physical significance. However, in Eq. (13), the parameters are of known characteristics of the materials and can be obtained from any standard book. Also, the formula checks the test curve with sufficient accuracy.

RECOMMENDATIONS

The recommendations are made that allowables for beams subjected to bending loads be given in the form of curves of norstress vs. relative dimensions or maximum fiber stresses for all shapes and materials. These curves should be based on a theoretical development such as outlined in this paper. However, experimental verification should be obtained as soon as possible.

For the present, the ANC-5 curves for calculating allowable values for tubings subjected to bending may be redrawn into curves with D/t as abscissas and with M/R^3 as ordinates instead of bending modulus of rupture. By doing this the norstress values can be obtained merely by dividing the bending modulus of rupture by

$$\left(\frac{1}{\pi}\right)\left(\frac{D}{t}\right)\frac{1}{[1 + (1 - 2t/D)^2][1 - (t/D)]}$$

APPENDIX A

Many stress-strain formulas have been developed.[9] Some of them contain one or two parameters and some more. However, the formula should be as general as possible so that it can be adaptable to many different materials. Also, the formula should be as simple as possible; that is, the parameters should be few in

number and easy to obtain. According to these points, Eq. (13) would seem to be preferable. To illustrate the value of this formula, a comparison with the experimental data is shown in Fig. 11.

Fig. 11 shows that Eq. (13) and the test curve up to strain equal to 0.0095 have excellent agreement. No test data are available to the author at strains above 0.0095, but, since the calculated value of stress at $\epsilon_{max.}$ is 87,700 lbs. per sq.in. against a test value of 88,000 lbs. per sq.in., the curves should be sufficiently accurate between $\epsilon = 0.0095$ and $\epsilon_{max.}$.

Appendix B

Eqs. (11) and (12) can be integrated mathematically by substituting the stress-strain formula, Eq. (13), into Eqs. (11) and (12) with the aid of Eqs. (a) and (b).

Substitution of Eq. (13) into Eqs. (a) and (b) yields

$$\int_0^{\sigma_1} \sigma d\epsilon = \frac{{\sigma_1}^2}{2E} + \epsilon_{1\ max.}\left(\sigma_1 - \frac{\sigma_{1\ ult.}}{K_1}\right)e^{K_1[(\sigma_1/\sigma_{1\ ult.})\ -\ 1]} + \frac{\epsilon_{1\ max.}\sigma_{1\ ult.}}{K_1 e^{K_1}} \quad (c)$$

$$\int_0^{\sigma_1} \sigma\epsilon d\epsilon = \frac{{\sigma_1}^3}{3E} + \frac{\epsilon_{1\ max.}}{EK_1^2}(K_1^2\sigma_1^2 - K_1\sigma_1\sigma_{1\ ult.} + {\sigma_1}^2_{ult.})\, e^{K_1[(\sigma_1/\sigma_{1\ ult.})\ -\ 1]} + \frac{{\epsilon_{1\ max.}}^2}{4K_1}(2K_1\sigma_1 - \sigma_{1\ ult.}) \times e^{2K_1[(\sigma_1/\sigma_{1\ ult.})\ -\ 1]} + \frac{\epsilon_{1\ max.}\sigma_{1\ ult.}}{K_1^2 e^{K_1}}\left(\frac{K_1\epsilon_{1\ max}}{4e^{K_1}} - \frac{\sigma_{1\ ult.}}{E}\right) \quad (d)$$

For simplicity, let

$$\phi(\sigma_1) = \int_0^{\sigma_1} \sigma d\epsilon, \qquad \theta(\sigma_1) = \int_0^{\sigma_1} \sigma\epsilon d\epsilon \quad (e)$$

Then,

$$\phi(\sigma_2) = \int_0^{-\sigma_2} \sigma d\epsilon, \qquad \theta(\sigma_2) = \int_0^{-\sigma_2} \sigma\epsilon d\epsilon \quad (f)$$

where $\phi(\sigma_2)$ and $\theta(\sigma_2)$ have the same expressions as $\phi(\sigma_1)$ and $\theta(\sigma_1)$, respectively, except that in the latter σ_2, ϵ_2, and K_2 are substituted for σ_1, ϵ_1, and K_1, respectively. With the aid of Eqs. (11) and (12), the stress equation and moment equation can be shown as follows:

$$\phi(\sigma_1) = \phi(\sigma_2) \quad (g)$$

and

$$\frac{1}{(\epsilon_1 + \epsilon_2)^2}[\theta(\sigma_1) + \theta(\sigma_2)] = \frac{M}{BH^2} \quad (h)$$

Appendix C—Tabular Forms

In view of the practical application of the stress and moment equations, Eqs. (11) and (12), the following tabular forms are recommended for using the graphical method. This part may be considered as a supplement to the section "Procedure for Calculating the Stress Distribution."

Tabular forms for rectangular section:

(1) See Table 1.

Table 1

ϵ_1	Assume some proper values
$\int_0^{\epsilon_1} \sigma d\epsilon$	Measure the corresponding areas from the σ-ϵ tension curve
ϵ_2	Assume some proper values
$\int_0^{-\epsilon_2} \sigma d\epsilon$	Measure the corresponding areas from the σ-ϵ compression curve

(2) Plot a curve with ϵ_1 against $\int_0^{\epsilon_1} \sigma d\epsilon$ and a curve with ϵ_2 against $\int_0^{-\epsilon_2} \sigma d\epsilon$ by using the above table.

(3) Plot a curve with ϵ_1 as abscissas and with ϵ_2 as ordinates. The ϵ_1 and ϵ_2 are obtained from the curves in item (2) and satisfied in Eq. (11).

(4) Plot a curve with ϵ as abscissas and $\sigma\epsilon$ as ordinates.

(5) See Table 2.

Table 2

1	ϵ_1	Assumed
2	$\int_0^{\epsilon_1} \sigma\epsilon d\epsilon$	Measure the corresponding areas from the $\sigma\epsilon_1 - \epsilon_1$ curve
3	ϵ_2	Values corresponding to ϵ_1 in (1) from the $\epsilon_1 - \epsilon_2$ curve
4	$\int_0^{-\epsilon_2} \sigma\epsilon d\epsilon$	Measure the corresponding areas from the $\sigma\epsilon_2 - \epsilon_2$ curve
5	$\int_0^{\epsilon_1} \sigma\epsilon d\epsilon + \int_0^{-\epsilon_2} \sigma\epsilon d\epsilon$	(2) + (4)
6	$\epsilon_1 + \epsilon_2$	(1) + (3)
7	$(\epsilon_1 + \epsilon_2)^2$	(6)^2
8	$[1/(\epsilon_2 + \epsilon_2)^2]$	1/(7)
9	M/bH^2	(5) × (8)
10	σ_1	Corresponding to ϵ_1
11	σ_2	Corresponding to ϵ_2

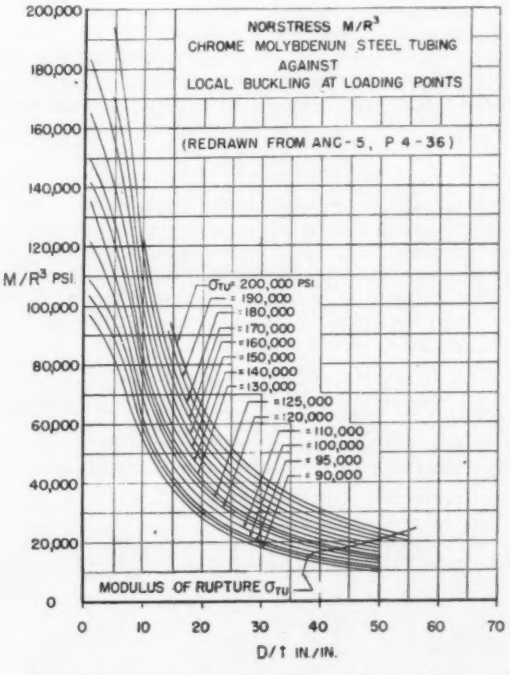

Fig. 12a. Norstress curves for alloy steel at ultimate load.

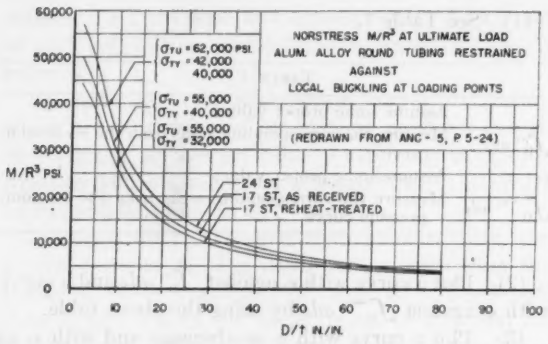

FIG. 12b. Norstress curves for aluminum alloy at ultimate load.

(6) Plot a curve with σ_1 and σ_2 as abscissas and M/BH^2 as ordinates.

APPENDIX D

For design use at the present, the curves of bending modulus of rupture for both chrome molybdenum steel tubing and aluminum-alloy tubing shown on pages 4–36 and 5–24 of ANC-5 have been redrawn into curves of norstress M/R^3 against D/t in Figs. 12.

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Letter to the Editor

Dear Sir:

The recent article by E. Arthur Bonney ("Aerodynamic Characteristics of Rectangular Wings at Supersonic Speeds," JOURNAL OF THE AERONAUTICAL SCIENCES, February, 1947) may be improved in accuracy by certain relatively minor changes.

The relation for drag coefficient was obtained by analogy with the derivation of lift coefficient. Actually, the effect of airfoil thickness or thickness distribution for symmetrical airfoils cancels out in the average section lift coefficient so that the lift coefficient obtained by the second-order approximation is identical with that obtained from the first-order approximation:

$$C_L = 4\alpha/\sqrt{M^2 - 1}$$

On the other hand, if the drag coefficient is set up as a function of the airfoil thickness and thickness distribution, the second-order terms do not cancel out. For example, consider a wedge-type airfoil of thickness ratio τ; a nondimensional chordwise location of maximum thickness X; β_1, the nose semiangle; and β_2, the trailing semiangle. Then $\beta_1 = \tau/2X$; $\beta_2 = \tau/2(1 - X)$; and the drag coefficient becomes

$$C_D = \frac{C_1\tau^2}{2}\left(\frac{1}{X} + \frac{1}{1-X}\right) + \frac{C_2\tau^2}{4}\left(\frac{1}{X^2} - \frac{1}{(1-X)^2}\right)$$

Differentiation with respect to X yields the relation between τ and the optimum chordwise location of maximum thickness:

$$\tau = -\frac{C_1}{C_2}\left[\frac{(1-2X)(X)(1-X)}{(1-3X+3X^2)}\right]$$

or approximately

$$X = 0.5[1 + (C_2/C_1)\tau]$$

The corresponding optimum drag coefficient becomes

$$C_{D_{opt.}} = 2C_1\tau^2[1 - (C_2^2\tau^2/C_1^2)]$$

For the special case of $X = 0.5$ (i.e., the doubly symmetric wedge airfoil) the drag varies as τ^2 as stated in conclusion (6) of the article but not in the optimum case. For the case of finite aspect ratios the thickness distribution at the tips should probably differ from that near the center section.

Conclusion (7) of the article says that the airfoil of minimum drag for a given thickness ratio or stress is symmetrical about both the chord line and midpoint. The above type of analysis indicates that the airfoil of minimum wave drag (at zero lift) for a given thickness ratio is a wedge-type airfoil symmetrical about the chord line but not symmetrical in a fore-and-aft direction. At an angle of attack the optimum airfoil for a given thickness is asymmetrical in both directions. If the thickness ratio is allowed to vary, then the airfoil of minimum pressure drag for a given structural strength will be some curved shape. Recent theoretical and experimental work indicates that the viscous drag (for certain Reynolds and Mach numbers) may be critically influenced by changes in airfoil shape so that the relative total drags (viscous plus wave) may be quite different from the relative wave drags. It is then possible for some curved airfoil to have less total drag than a wedge-type airfoil of the same thickness ratio.

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